Saudi Journal of Civil Engineering
A Publication by "Scholars Middle East Publishers", Dubai, United Arab Emirates

ISSN 2523-2657 (Print)
ISSN 2523-2231 (Online)

Picard's Successive Iteration Method for the Elastic Buckling Analysis of Euler Columns with Pinned Ends

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Article History
Received: 04.09.2018
Accepted: 15.09.2018
Published: 30.10.2018

Abstract: In this work, the Picard's successive iteration method was used to solve the elastic buckling problem of Euler columns with pinned ends. The problem was represented by a second order ordinary differential equation in the deflection function subject to the boundary conditions at the pinned ends. The boundary value problem was expressed in integral form, and the Picard's iteration scheme developed from the integral form. A suitable buckling shape function was used to obtain an initial approximation to the deflection, and the Picard's iteration scheme used to obtain first, second and third iterations for the modal buckling functions through the use of corresponding boundary conditions. Corresponding Picard's approximations for the first, second and third iterations of the critical buckling load were obtained as $P_{cr}^{(1)} = 9.60EI/l^2$, $P_{cr}^{(2)} = 9.8361EI/l^2$ and $P_{cr}^{(3)} = 9.8657EI/l^2$. The errors in the first, second and third iterates of the critical buckling load were -2.732%, -0.339% and -0.040% respectively. The use of the exact buckling shape function in the Picard's iteration scheme was found to result in the exact closed form solution for the critical buckling load.

Keywords: Picard's successive iteration method, critical buckling load, elastic buckling problem, Euler column, buckling shape function

INTRODUCTION

Columns are long slender structural members subject to axial compressive forces. They are classified according to their slenderness ratio as short, intermediate or long columns; and can fail by buckling [1-4]. Buckling failures are characterised by geometric instabilities where the lateral displacement of the axially compressed column suddenly becomes excessive [5-8]. There are two basic types of buckling, namely: elastic buckling and non-elastic buckling. Elastic buckling failure takes place when the material yield strength of the column has not been attained [9]. Elastic buckling is characterised by a sudden failure of the column subject to axial compressive forces where the ultimate yield strength of the column material has not been attained [9, 10].

Buckling problems are formulated and solved as linear buckling analysis problem (eigen value eigen vector problems) or as non-linear buckling problems (plastic or non-linear elastic buckling problems) [11]. The specific objective of linear elastic buckling analysis is to determine the buckling loads and the critical buckling load, and this usually entails solving an eigen vector eigen value problem in mathematical/engineering analysis. Literature review shows that the methods of finding the critical buckling loads in elastic buckling problems are classified into two namely: exact mathematical or analytical methods and approximate or numerical methods.

The exact mathematical methods involve all the methods available in mathematics for solving the governing differential equations of equilibrium subject to their boundary conditions, and usually yield exact expressions for the eigen vectors and the eigen values of the boundary value problem (BVP). They include D – operator methods, eigen function expansion methods, integral transform methods, method of trial functions, variation of parameters, etc.

The approximate or numerical methods seek to obtain approximations of the exact solutions for the eigen vectors and eigen values using energy principles, variational techniques or discrete approximations of the governing differential equations of equilibrium and the associated boundary conditions.

The exact mathematical methods are difficult to solve in closed form except for simple elastic column buckling problems. Approximate methods are used to solve column buckling problems that are not easily solved by the exact mathematical methods. Approximate methods have been used to solve the column buckling problem by Zdravkovic *et al*., [12], Li *et al*., [13], Huang and Li [14], Kalakowski *et al*., [15], Reddy [16], Yaun and Wang [17], Atay [18], Okay *et al*., [19] and Ofondu *et al*., [20].

Ofondu *et al*., [20] used the Bubnov-Galerkin variational method to present a one – parameter and two – parameter variational formulation of the elastic buckling problem of Euler columns. They then solved the elastic buckling problem of Euler – columns with fixed pinned ends using one and two parameter shape functions in the Bubnov-Galerkin variational method. In each case, the Bubnov-Galerkin method reduced the BVP to an algebraic eigenvalue eigenvector problem. The solution of the characteristic homogeneous equations yielded buckling loads which agreed with classical mathematical solutions.

Basbük *et al*., [21] used the Homotopy analysis method (HAM) to solve the critical buckling load problem of a column under end load dependent on direction. Eryilmaz *et al*., [11] used the Homotopy analysis method (HAM) to solve the buckling problem of Euler columns with continuous elastic restraint. Atay [18] used the Homotopy perturbation method (HPM) to solve the critical buckling load problem for Euler columns with variable stiffness. Okay *et al*., [19] used the variational iteration method (VIM) to obtain buckling loads and buckling modal shape functions for columns. Yuan and Wang [17] used the differential quadrature method (DQM) to perform buckling and post buckling analysis of beam columns. Reddy [16] used the finite element method (FEM) to perform buckling analysis of cracked stepped column. Zdravkovic *et al*., [12] efficiently estimated the elastic buckling load of axially compressed three segment stepped column using the energy method.

RESEARCH AIM AND OBJECTIVES

In this work, the Picard's successive iteration method is used to determine the critical buckling load of prismatic Euler column of length l, with simply supported ends. The specific objectives include:

- To express the governing ordinary differential equation (ODE) of elastic buckling of Euler column in Picard's iteration form using the integral form of the ODE.
- To obtain an initial approximation of the buckled mode shape function satisfying the boundary conditions of simple supports at the ends.
- To perform the Picard's iteration scheme and obtain expressions for the Picard's first approximation to the buckling mode shapes and corresponding Picard's first approximation to the buckling load.
- To perform successive Picard's iterations and obtain the Picard's second, and third buckling mode shapes and corresponding buckling loads.
- To use the Picard's iteration method and obtain closed form mathematical solutions for the buckling mode shapes and buckling loads, and critical buckling load.

Theoretical framework

Theory of elastic buckling of Euler columns

Assumptions of Euler's theory of column buckling

The fundamental assumptions of the Euler's theory of elastic buckling of columns are [1, 5, 6, 21, 22]:

- The column is straight in the longitudinal coordinate axis before the axial compressive load is applied.
- The cross-section of the column is uniform along the longitudinal axis.
- The material is homogeneous and isotropic.
- The self weight is neglected.
- Failure is due to buckling alone.
- The reduction in the length of the column is very small and negligible.

Derivation of governing equations

The axial, shear forces and bending moments that act on an element of a column that is under axially compressive forces and transverse load $q(x)$ are shown in Figure-1.

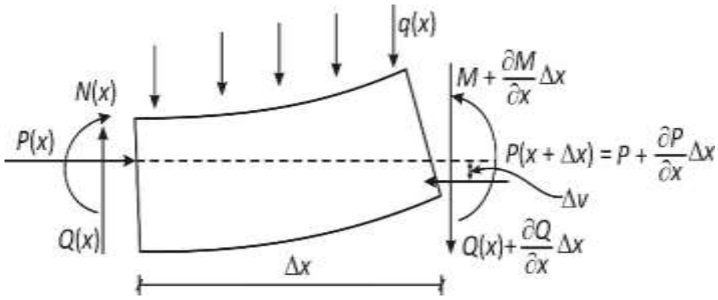

Fig-1: Forces acting on an element of a column

The governing ordinary differential equation for column buckling is derived following a similar derivation principle as that for bending beams but including the effect of axial forces [1, 3, 22]. For equilibrium of forces in the horizontal direction

$$\sum \cdot F_{ix} = P(x) + \frac{\partial P}{\partial x}\Delta x - P(x) = 0 \qquad (1)$$

$$\frac{\partial P}{\partial x}\Delta x = 0 \qquad (2)$$

$\therefore$ $P(x)$ is constant.

For equilibrium of forces in the vertical direction,

$$Q(x) + \frac{\partial Q(x)}{\partial x}\Delta x - Q(x) + q(x)\Delta x = 0 \qquad (3)$$

$$\frac{\partial Q(x)}{\partial x} = -q(x) \qquad (4)$$

For equilibrium of forces in rotation,

$$M(x) + \frac{\partial M(x)}{\partial x}\Delta x - M(x) + q(x)\Delta x\frac{\Delta x}{2} - Q(x)\Delta x - P\Delta v = 0 \qquad (5)$$

$$\frac{\partial M(x)}{\partial x}\Delta x + q(x)\frac{(\Delta x)^2}{2} - Q(x)\Delta x - P\Delta v = 0 \qquad (6)$$

$$\frac{\partial M}{\partial x} + q(x)\frac{\Delta x}{2} - Q(x) - \frac{P\Delta v}{\Delta x} = 0 \qquad (7)$$

$$\frac{\partial M}{\partial x} - P\frac{\Delta v}{\Delta x} = Q(x) - q(x)\frac{\Delta x}{2} \qquad (8)$$

As $\Delta x \to 0$, we have

$$\frac{dM}{dx} - P\frac{dv}{dx} = Q(x) \qquad (9)$$

By differentiation of Equation (9) with respect to x, we obtain:

$$\frac{d}{dx}\left(\frac{dM}{dx} - P\frac{dv}{dx}\right) = \frac{dQ(x)}{dx} = -q(x) \qquad (10)$$

Simplifying,

$$\frac{d^2M}{dx^2} - P\frac{d^2v}{dx^2} = -q(x) \qquad (11)$$

The bending moment – curvature relation for Euler – Bernoulli beam theory, is:

$$M = -EI\frac{d^2v}{dx^2} \qquad (12)$$

Equation (11) then becomes:

$$\frac{d^2}{dx^2}\left(-EI\frac{d^2v}{dx^2}\right) - P\frac{d^2v}{dx^2} = -q(x) = \frac{d^2}{dx^2}\left(-EI\frac{d^2v}{dx^2} - Pv\right) = -q(x) \qquad (13)$$

For prismatic cross-sections, *EI* is constant, and we obtain: [1]

$$-EI\frac{d^4v}{dx^4} - P\frac{d^2v}{dx^2} = -q(x) \qquad (14)$$

or $EI\frac{d^4v(x)}{dx^4} + P\frac{d^2v(x)}{dx^2} = q(x)$ (15)

Equation (15), a fourth order ordinary differential equation in terms of $v(x)$ governs elastic buckling of Euler columns.

METHODOLOGY

Picard's successive iteration method

Picard's successive iteration method is based on the idea of generating a sequence of approximate solutions to an initial value problem (IVP) or boundary value problem (BVP) from an initial function that is made to satisfy given boundary conditions of the problem. The initial approximation of the solution is then improved by successive integrations of the governing differential equation, and thus a sequence of approximate solutions can be obtained by this method. For cases involving ordinary differential equations (ODE) the sequence of approximate functions (solutions) converges to the exact solution when the initial function satisfies the boundary conditions and is integrable, and continuous [23-30].

For the initial value problem IVP,

$$y' = f(t, y), \quad y(t_0) = y_0 \qquad (16)$$

Where, $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous in some region around the point (x_0, y_0), the Picard's iteration method converts the IVP to the integral equation [23-30]

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds \qquad (17)$$

The integral formulation is then used to construct a sequence of approximate solutions as follows:

$$y_k(t) = y_0 + \int_{t_0}^{t} f(s, y_{k-1}(s))\,ds \qquad (18)$$

The basic idea is that given an initial approximation to the IVP, say $y_0(x)$, an infinite sequence of functions $y_n(x)$ is constructed following the iteration scheme [23 – 30].

RESULTS

Picard's first iteration

The elastic buckling problem of Euler column to be solved is given by the second order ordinary differential equation which is a special case of Equation (13):

$$EI\,v''(x) = -Pv(x) \qquad (19)$$

Let

$$v(x) = v_1(x) \qquad (20)$$

Then

$$v_2(x) = -P\int_0^x\int_0^x \frac{v_1(x)}{EI}\,dx\,dx \qquad (21)$$

An Euler column of length, l with pinned ends is considered. If the origin of the Cartesian coordinates is chosen at the centre of the pinned – pinned column, then the boundary conditions are:

$$v(x = \pm l/2) = 0 \qquad (22)$$

$$v''(x = \pm l/2) = 0 \qquad (23)$$

A suitable shape (basis or coordinate) function is thus given by

$$\varphi(x) = 1 - \left(\frac{x}{l/2}\right)^2 = 1 - \left(\frac{2x}{l}\right)^2 \qquad (24)$$

$$\varphi(x) = 1 - \frac{4x^2}{l^2} \qquad (25)$$

Thus, let

$$v_1(x) = v_0\left(1 - \left(\frac{2x}{l}\right)^2\right) \qquad (26)$$

Then,

$$v_2''(x) = \frac{-Pv_1(x)}{EI} = \frac{-Pv_0}{EI}\left(1 - \left(\frac{2x}{l}\right)^2\right) \qquad (27)$$

By integration,

$$v_2'(x) = \int_0^x v_2''(x)\,dx = \int_0^x \frac{-Pv_0}{EI}\left(1 - \left(\frac{2x}{l}\right)^2\right)dx \qquad (28)$$

$$v_2'(x) = \frac{-Pv_0}{EI}\int_0^x \left(1 - \left(\frac{2x}{l}\right)^2\right)dx \qquad (29)$$

$$v_2'(x) = \frac{-Pv_0}{EI}\int_0^x \left(1 - \frac{4x^2}{l^2}\right)dx \qquad (30)$$

$$v_2'(x) = \frac{-Pv_0}{EI}\left(x - \frac{4x^3}{3l^2} + c_1\right) \qquad (31)$$

For symmetry,

$$v_2'(x = 0) = 0 \qquad (32)$$

$$\therefore\; v_2'(x = 0) = \frac{-Pv_0}{EI}(0 + c_1) = 0 \qquad (33)$$

$$c_1 = 0 \qquad (34)$$

Hence,

$$v_2'(x) = \frac{-Pv_0}{EI}\left(x - \frac{4x^3}{3l^2}\right) \quad (35)$$

Integrating again,

$$v_2(x) = \int_0^x v_2'(x)\,dx = \int_0^x \frac{-Pv_0}{EI}\left(x - \frac{4x^3}{3l^2}\right)dx \quad (36)$$

$$v_2(x) = \frac{-Pv_0}{EI}\int_0^x \left(x - \frac{4x^3}{3l^2}\right)dx \quad (37)$$

$$= \frac{-Pv_0}{EI}\left(\frac{x^2}{2} - \frac{4x^4}{12l^2} + c_2\right) \quad (38)$$

$$= \frac{-Pv_0}{EI}\left(\frac{x^2}{2} - \frac{x^4}{3l^2} + c_2\right) \quad (39)$$

From the boundary condition,

$$v_2(x = \pm l/2) = 0 \quad (40)$$

$$v_2(x = l/2) = \frac{-Pv_0}{EI}\left(\frac{1}{2}\left(\frac{l}{2}\right)^2 - \frac{1}{3l^2}\left(\frac{l}{2}\right)^4 + c_2\right) = 0 \quad (41)$$

$$v_2(x = l/2) = \frac{-Pv_0}{EI}\left(\frac{l^2}{8} - \frac{l^4}{48l^2} + c_2\right) = 0 \quad (42)$$

$$c_2 = \frac{-5l^2}{48} \quad (43)$$

$$v_2(x) = \frac{-Pv_0}{EI}\left(\frac{-5l^2}{48} + \frac{x^2}{2} - \frac{x^4}{3l^2}\right) \quad (44)$$

$$v_2(x) = \frac{5Pl^2v_0}{48EI}\left(1 - \frac{48}{5l^2}\frac{x^2}{2} + \frac{48}{5l^2}\frac{x^4}{3l^2}\right) \quad (45)$$

$$v_2(x) = \frac{5Pl^2v_0}{48EI}\left(1 - \frac{24x^2}{5l^2} + \frac{16}{5}\frac{x^4}{l^4}\right) \quad (46)$$

Picard's first approximation for the buckling load

Picard's first approximation to the buckling load is found by considering

$$v_2(x = 0) = v_1(x = 0) \quad (47)$$

Then,

$$v_2((0) = \frac{5Pl^2v_0}{48EI} = v_0\left(1 - \frac{4x^2}{l^2}\right)\Bigg|_{x=0} = v_0 \quad (48)$$

$$P^{(1)} = \frac{48}{5}\frac{EI}{l^2} = 9.60\frac{EI}{l^2} \quad (49)$$

The exact buckling load for pinned-pinned Euler column is

$$P_{\text{exact}} = \pi^2\frac{EI}{l^2} = 9.8696\frac{EI}{l^2} \quad (50)$$

Picard's second iteration

$$EI\, v_3''(x) = -P^{(2)} v_2(x) \qquad (51)$$

$$v_3''(x) = \frac{-P^{(2)}}{EI} v_2(x) = \frac{-P^{(2)}}{EI} \cdot \frac{5P^{(1)} l^2 v_0}{48EI} \left(1 - \frac{24x^2}{5l^2} + \frac{16x^4}{5l^4}\right) \qquad (52)$$

Integrating with respect to x,

$$v_3'(x) = \int_0^x v_3''(x)\,dx = \int_0^x \frac{-P^{(2)}}{EI} \frac{5P^{(1)} l^2 v_0}{48EI} \left(1 - \frac{24x^2}{5l^2} + \frac{16x^4}{5l^4}\right) dx \qquad (53)$$

$$v_3'(x) = \frac{-P^{(2)}}{EI} \cdot \frac{5P^{(1)} l^2 v_0}{48EI} \left(x - \frac{24x^3}{15l^2} + \frac{16x^5}{25l^4} + c_3\right) \qquad (54)$$

From symmetry considerations,

$$v_3'(x = 0) = 0 \qquad (55)$$

$$v_3'(0) = \frac{-P^{(2)}}{EI} \cdot \frac{5P^{(1)} l^2 v_0}{48EI} (0 + c_3) = 0 \qquad (56)$$

$$c_3 = 0 \qquad (57)$$

Hence,

$$v_3'(x) = \frac{-P^{(2)}}{EI} \frac{5P^{(1)} l^2 v_0}{48EI} \left(x - \frac{24x^3}{15l^2} + \frac{16x^5}{25l^4}\right) \qquad (58)$$

Integrating with respect to x,

$$v_3(x) = \int_0^x v_2'(x)\,dx = \int_0^x \frac{-P^{(2)}}{EI} \frac{5P^{(1)} l^2 v_0}{48EI} \left(x - \frac{24x^3}{15l^2} + \frac{16x^5}{25l^4}\right) dx \qquad (59)$$

$$v_3(x) = \frac{-P^{(2)}}{EI} \frac{5P^{(1)} l^2 v_0}{48EI} \left(\frac{x^2}{2} - \frac{24x^4}{60l^2} + \frac{16x^6}{150l^4} + c_4\right) \qquad (60)$$

$$v_3(x) = \frac{-P^{(2)}}{EI} \frac{5P^{(1)} l^2 v_0}{48EI} \left(\frac{x^2}{2} - \frac{2}{5}\frac{x^4}{l^2} + \frac{8x^6}{75l^4} + c_4\right) \qquad (61)$$

From the boundary condition

$$v_3\left(x = \frac{l}{2}\right) = 0 \qquad (62)$$

We have,

$$\frac{-P^{(2)}}{EI} \frac{5P^{(1)} l^2 v_0}{48EI} \left(\frac{1}{2}\left(\frac{l}{2}\right)^2 - \frac{2}{5l^2}\left(\frac{l}{2}\right)^4 + \frac{8}{75l^4}\left(\frac{l}{2}\right)^6 + c_4\right) = 0 \qquad (63)$$

$$\frac{-P^{(2)}}{EI} \frac{5P^{(1)} l^2 v_0}{48EI} \left(\frac{l^2}{8} - \frac{l^2}{40} + \frac{l^2}{600} + c_4\right) = 0 \qquad (64)$$

$$c_4 = \frac{-61l^2}{600} \qquad (65)$$

Hence,

$$v_3(x) = \frac{-P^{(2)}}{EI} \frac{5P^{(1)} l^2 v_0}{48EI} \left(\frac{x^2}{2} - \frac{2}{5}\frac{x^4}{l^2} + \frac{8x^6}{75l^4} - \frac{61l^2}{600}\right) \qquad (66)$$

Alternatively,

$$v_3(x) = \frac{-61l^2}{600}\frac{-P^{(2)}}{EI}\frac{5P^{(1)}l^2v_0}{48EI}\left(1 - \frac{600}{61l^2}\frac{x^2}{2} + \frac{600}{61l^2}\left(\frac{2x^4}{5l^2}\right) - \frac{600}{61l^2}\frac{8x^6}{75l^4}\right) \quad (67)$$

$$v_3(x) = \frac{61}{5760}\frac{P^{(2)}P^{(1)}l^4v_0}{EI\cdot EI}\left(1 - \frac{300}{61}\left(\frac{x}{l}\right)^2 + \frac{240}{61}\left(\frac{x}{l}\right)^4 - \frac{64}{61}\left(\frac{x}{l}\right)^6\right) \quad (68)$$

Picard's second approximation to the buckling load

Picard's second approximation $P^{(2)}$ to the elastic buckling load is found by considering

$$v_3(x=0) = v_2(x=0) \quad (69)$$

$$\frac{61}{5760}\frac{P^{(2)}P^{(1)}l^4v_0}{EI\cdot EI}(1) = \frac{5P^{(1)}l^2v_0}{48EI}(1) \quad (70)$$

$$P^{(2)} = \frac{5760EI\,EI}{61P^{(1)}l^4v_0}\cdot\frac{5P^{(1)}l^2v_0}{48EI} \quad (71)$$

$$P^{(2)} = \frac{5760}{61}\times\frac{5}{48}\frac{EI}{l^2} = \frac{600}{61}\frac{EI}{l^2} \quad (72)$$

$$P^{(2)} = 9.8361\frac{EI}{l^2} \quad (73)$$

Picard's third iteration

The Picard third iteration is given by

$$EI\,v_4''(x) = -P^{(3)}v_3(x) \quad (74)$$

$$v_4''(x) = \frac{-P^{(3)}}{EI}v_3(x) \quad (75)$$

$$= \frac{-P^{(3)}}{EI}\left(\frac{61P^{(2)}P^{(1)}l^4v_0}{5760(EI)^2}\right)\left(1 - \frac{300}{61}\frac{x^2}{l^2} + \frac{240}{61}\frac{x^4}{l^4} - \frac{64}{61}\frac{x^6}{l^6}\right) \quad (76)$$

Integrating with respect to x,

$$v_4'(x) = \int_0^x v_4''(x)\,dx \quad (77)$$

$$= \frac{-P^{(3)}}{EI}\frac{61P^{(2)}P^{(1)}l^4v_0}{5760(EI)^2}\int_0^x\left(1 - \frac{300x^2}{61l^2} + \frac{240x^4}{61l^4} - \frac{64x^6}{61l^6}\right)dx \quad (78)$$

$$v_4'(x) = \frac{-P^{(3)}61P^{(2)}P^{(1)}l^4v_0}{5760(EI)^3}\left(x - \frac{300x^3}{61\times 3l^2} + \frac{240}{61}\frac{x^5}{5l^4} - \frac{64}{61}\left(\frac{x^7}{7l^6}\right) + c_5\right) \quad (79)$$

From symmetry about $x = 0$,

$$v_4'(x=0) = 0 \quad (80)$$

$$\therefore\ c_5 = 0 \quad (81)$$

Hence,

$$v_4'(x) = \frac{-P^{(3)}61}{5760}\frac{P^{(2)}P^{(1)}l^4v_0}{(EI)^3}\left(x - \frac{300x^3}{183l^2} + \frac{240x^5}{305l^4} - \frac{64x^7}{427l^6}\right) \quad (82)$$

By integration,

$$v_4(x) = \int_0^x v_4'(x)\,dx = \frac{-61}{5760}\frac{P^{(3)}P^{(2)}P^{(1)}l^4 v_0}{(EI)^3}\int_0^x \left(x - \frac{300}{61}\frac{x^3}{3l^2} + \frac{240}{61}\frac{x^5}{5l^4} - \frac{64}{61}\frac{x^7}{7l^6} \right) dx \quad (83)$$

$$v_4(x) = \frac{-61}{5760}\frac{P^{(3)}P^{(2)}P^{(1)}l^4 v_0}{(EI)^3}\left(\frac{x^2}{2} - \frac{310}{61}\frac{x^4}{12l^2} + \frac{240}{61}\frac{x^6}{30l^4} - \frac{64}{61}\frac{x^8}{56l^6} + c_6 \right) \quad (84)$$

Using the boundary condition,

$$v_4\left(x = \frac{l}{2}\right) = 0 \quad (85)$$

$$\frac{-61}{5760}\frac{P^{(3)}P^{(2)}P^{(1)}l^4 v_0}{(EI)^3}\left(\frac{1}{2}\left(\frac{l}{2}\right)^2 - \frac{300}{61\times 12l^2}\left(\frac{l}{2}\right)^4 + \frac{240}{61\times 30l^4}\left(\frac{l}{2}\right)^6 - \frac{64}{61\times 56l^6}\left(\frac{l}{2}\right)^8 + c_6 \right) = 0 \quad \ldots(86)$$

$$c_6 = -0.101361241l^2 \quad (87)$$

$$v_4(x) = \frac{-61}{5760}\frac{P^3P^2P^1l^4v_0}{(EI)^3}\left(\frac{x^2}{2} - \frac{300x^4}{732l^2} + \frac{240x^6}{1830l^4} - \frac{64x^8}{3416l^6} - 0.101361241l^2 \right) \quad (88)$$

Alternatively

$$v_4(x) = \frac{-61}{5760}\frac{(0.101361241l^2)P^3P^2P^1l^4v_0}{(EI)^3}\left(-1 + \frac{9.86503971x^2}{2} - \frac{300\times 9.86503971x^4}{732l^2} + \frac{240\times 9.865703971x^6}{1830l^4} - \frac{64\times 9.865703971x^8}{3416l^6} \right) \quad (89)$$

$$v_4(x) = \frac{-6.183035701P^3P^2P^1l^6v_0}{5760(EI)^3}\left(-1 + \frac{9.86503971x^2}{2} - \frac{300\times 9.86503971x^4}{732l^2} + \frac{240(9.865703971x^6)}{1830l^4} - \frac{64\times 9.865703971x^8}{3416l^6} \right) \quad (90)$$

Picard’s third approximation to the elastic buckling load

Picard’s third approximation to the elastic buckling load is found by considering

$$v_4(x = 0) = v_3(x = 0) \quad (91)$$

$$\frac{-6.183035701P^{(3)}P^{(2)}P^{(1)}l^6v_0}{5760(EI)^3}(-1) = \frac{61}{5760}\frac{P^{(2)}P^{(1)}l^4v_0}{(EI)^2} = \frac{6.183035701P^3P^2P^1l^6v_0}{5760(EI)^3} \quad (92)$$

$$P^{(3)} = \frac{5760(EI)^3}{6.183035701}\cdot\frac{61P^{(2)}P^{(1)}l^4v_0}{P^{(2)}P^{(1)}l^6v_0 5760(EI)^2} \quad (93)$$

$$P^{(3)} = \frac{61}{6.183035701}\left(\frac{EI}{l^2}\right) \quad (94)$$

$$P^{(3)} = 9.865703992\frac{EI}{l^2} \quad (95)$$

$$P^{(3)} = 9.8657\frac{EI}{l^2} \quad (96)$$

Picard's successive iteration method for obtaining closed form exact solution
The exact shape function for the *n*th modal buckling shape is given by

$$v_n(x) = v_0 \sin\frac{n\pi x}{l} \quad (97)$$

Where the origin is defined at one end of the column and the boundary conditions become

$$v_n(x = 0) = 0 \quad (98)$$

$$v_n''(x = 0) = 0 \quad (99)$$

$$v_n(x = l) = 0 \quad (100)$$

$$v_n''(x = l) = 0 \quad (101)$$

Then the Picard's iteration scheme becomes:

$$v_{n+1}''(x) = \frac{-P}{EI}v_0 \sin\frac{n\pi x}{l} \quad (102)$$

By successive integrations,

$$v_{n+1}'(x) = \frac{-Pv_0}{EI}\left(\frac{-l}{n\pi}\right)\cos\frac{n\pi x}{l} + a_1 \quad (103)$$

$$v_{n+1}(x) = \frac{-Pv_0}{EI}\left(-\left(\frac{l}{n\pi}\right)^2\right)\sin\frac{n\pi x}{l} + a_1 x + a_2 \quad (104)$$

Where,
a_1 and a_2 are the constants of integration.

Using the boundary conditions

$$v_{n+1}(x = 0) = 0 \quad (105)$$

We have

$$a_2 = 0 \quad (106)$$

From $v_{n+1}(x = l) = 0$ (107)

$$\frac{-P}{EI}\left(-\frac{l^2}{(n\pi)^2}\right)\sin n\pi + a_1 l = 0 \quad (108)$$

$$a_1 = 0 \quad (109)$$

Then,

$$v_{n+1}(x) = \frac{-Pv_0}{EI}\left(\frac{-l^2}{(n\pi)^2}\sin\frac{n\pi x}{l}\right) = \frac{Pl^2 v_0}{EI(n\pi)^2}\sin\frac{n\pi x}{l} \quad (110)$$

The *n*th buckling load is found from

$$v_{n+1}(x) = v_n(x) \quad (111)$$

$$\frac{Pl^2 v_0}{EI(n\pi)^2}\sin\frac{n\pi x}{l} = v_0 \sin\frac{n\pi x}{l} \quad (112)$$

$$\therefore \quad \frac{P l^2 v_0}{EI(n\pi)^2} = v_0 \qquad (113)$$

$$P = \frac{EI(n\pi)^2}{l^2} = (n\pi)^2 \frac{EI}{l^2} \qquad (114)$$

The lowest (critical) buckling mode is found when $n = 1$, and

$$P_{cr} = \pi^2 \frac{EI}{l^2} \qquad (115)$$

The critical buckling modal shape is given by

$$\varphi(x) = \sin\left(\frac{\pi x}{l}\right) \qquad (116)$$

DISCUSSION

In this paper, the elastic buckling problem of Euler columns has been successfully solved using the Picard's successive iteration method. The elastic buckling problem of Euler columns is mathematically expressed as an ordinary differential equation subject to boundary conditions which are defined by the conditions of support of the column at the ends. This work considered an Euler column with simple support conditions at $x = \pm l/2$ where the origin was considered at the centre from symmetry considerations. The shape (basis or coordinate) function that satisfies the boundary conditions (Equations (22) and (23)) was found as Equation (25) yielding the initial approximation of the deflection function as Equation (26) and the Picard's first iteration as Equation (27). By successive integration and use of boundary conditions, the Picard first iteration yielded Equation (46), resulting in the Picard's first approximation of the buckling load as Equation (49). The Picard's first approximation was found to be 2.732% lower than the exact value given as Equation (50). The Picard second iteration was given as Equation (52) and by successive integration and use of boundary conditions, the second iteration was obtained as Equation (68). The second approximation to the buckling load found using Equation (69) was expressed as Equation (73). The second approximation to the critical buckling load was found to be 0.339% lower than the exact critical buckling load. The Picard's third iteration was obtained using $v_3(x)$ as Equation (76). Successive integration with respect to x and the use of boundary conditions yielded the contents of integration and hence the third buckling mode as Equation (90). The third approximation to the elastic buckling load obtained by using Equation (91) was given as Equation (96). The third approximation to the critical buckling load was found to be 0.04% less than the exact buckling load.

An illustration of the application of the Picard's successive iteration method for solving the elastic buckling problem in closed form was also presented. For this, the exact buckling shape function for pinned-pinned Euler column for the nth modal buckling shape was used in Equation (97). The origin of Cartesian coordinates was redefined to coincide with one end of the Euler column yielding the boundary conditions given by Equations (98 – 101). The Picard's iteration scheme was thus obtained as Equation (102) for the (n +1)th iteration. Successive integration with respect to x and the use of the boundary conditions gave the unknown constants of integration as Equations (106) and (109) and the (n + 1)th modal buckling shape function as Equation (110). The nth buckling load was obtained using Equation (111) as Equation (114). The critical buckling load was found as Equation (115) and the critical buckling shape was found as Equation (116).

CONCLUSIONS

The conclusions of the present study are as follows:

- The Picard's successive iteration method is based on expressing the boundary value problem represented by an ordinary differential equation and boundary conditions as an integral equation and using the integral form to express the (n + 1)th iteration in terms of the nth iteration.
- At convergence the (n +1)th iteration will yield values sufficiently close to the nth iterates.
- The Picard's successive iteration method can be used to obtain closed form mathematical solutions to the elastic buckling problem of Euler columns with pinned ends by using exact buckling shape functions in the iteration scheme. The critical buckling load and buckling loads obtained in that case were exact.
- The Picard's first iteration for the shape function (Equation (25)) gave critical buckling load that was 2.732% lower than the exact critical buckling load. Similarly, the second and third approximations gave critical buckling loads that were 0.339% and 0.040% respectively less than the exact buckling load.

- The Picard's first approximation to the critical buckling load was just □2.732% in error, relative to the exact solution and the approximation is a reasonable estimate of the exact critical buckling load.
- The Picard's third approximation to the critical buckling load which was just □0.04% in error was sufficiently close to the exact critical buckling load and can be said to have converged.

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