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Least Squares Weighted Residual Method for the Elastic Buckling of Euler Column with Fixed-Pinned Ends

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Abstract: The least squares weighted residual method was used in this work to solve the boundary value problem (BVP) of an Euler column of length l fixed at $x = 0$, and pinned at $x = l$. Polynomial shape (spline) functions for Euler columns with fixed-pinned ends were used to obtain one – and two parameter buckling shape functions in terms of unknown generalised parameters. The one and two parameter buckling shape functions were used to construct least squares weighted residual integral statements of the boundary value problem. The least squares weighted residual statements simplified the boundary value problem (BVP) to algebraic eigenvalue – eigenvector problems. The solution for non trivial cases yielded characteristic buckling equations which were solved to obtain the buckling loads. One parameter coordinate shape function yielded the critical load as $Q_{cr} = 21EI/l^2$, while the two parameter buckling shape function yielded $Q_{cr} = 20.34614EI/l^2$. One parameter least squares weighted residual solution yielded a relative error of 4 % while the two parameter least squares weighted residual solution yielded a relative error of 0.77% compared to the exact solution.

Keywords: Least squares weighted residual method, elastic buckling, Euler column buckling, critical buckling load, buckling load, buckling mode.

INTRODUCTION

Background

Columns are common structural members in building and machine structures. Under compressive forces they are prone to failures called buckling. Elastic buckling problems are thus frequently encountered in the design of building and machine structures; and this entails the determination of critical loads at which buckling failures can occur for different types of end support conditions [1-6].

Two general approaches are used to solve elastic buckling problems, namely: analytical (or mathematical) and approximate (or numerical) methods [7-11]. The mathematical methods involve the methods available for solving the governing ordinary linear differential equations of equilibrium subject to the specific restraint conditions [1]. Mathematical methods usually yield mathematical solutions for the buckling loads, and buckling shapes of the boundary value problem. The mathematical methods are: Fourier series method, Laplace transform methods, D operator methods, variation of parameters, method of trial functions etc. In the approximate (or numerical) methods, the energy methods, variational principles and discretizations of the governing equations are used to obtain approximations to the buckling loads and buckling shapes of the problems. Some numerical methods include finite element method, finite difference method, weighted residual methods, collocation methods, etc. Approximate methods have the advantage that they can solve certain elastic buckling problems that cannot be easily solved using the exact analytical methods. Approximate methods have been applied to solve the column buckling problem, by various researchers, namely Zdravkovic *et al*., [12], Li *et al*., [13], Huang and Li [14], Kalakowski *et al*., [15], Reddy [16], Yuan and Wang [17], Atay [18], Okay *et al*., [19] and Ofondu *et al*., [20].

RESEARCH AIM AND OBJECTIVES

The research aim is to use the least squares weighted residual method to determine the critical buckling load of Euler columns with fixed pinned ends. The specific objectives are:

- To express the governing ordinary differential equation (ODE) of elastic buckling of Euler columns as least squares weighted residual integrals using one parameter buckling displacement modal shape functions that satisfy the boundary conditions at the fixed and pinned ends.
- To solve the one parameter and two parameter least squares weighted integral statements of the boundary value problem (BVP) of elastic buckling of Euler columns, and obtain the algebraic equivalent eigenvalue eigenvector problem for non-trivial solutions.
- To solve the algebraic eigenvalue eigenvector problem for non-trivial solutions and obtain the characteristic buckling equations for the one parameter and two parameter buckling shape functions.
- To solve the algebraic eigenvalue problem and obtain the critical buckling load for the one-parameter and two parameter buckling shape functions used.

Theory of elastic buckling

Assumptions

The study used the Euler's theory of elastic buckling of columns which is based on the following basic assumptions:

- The column's longitudinal coordinate axis is straight before the application of compressive force.
- The column cross-section is prismatic.
- The column is made of isotropic, homogeneous material.
- Column self weight is disregarded.
- Failure of the column is as a result of buckling only.
- Decrease in the longitudinal dimension of the column is small and can be disregarded.

Governing equations

The free body diagram of an infinitesimal segment of an Euler column under axial compression is as shown in Figure-1.

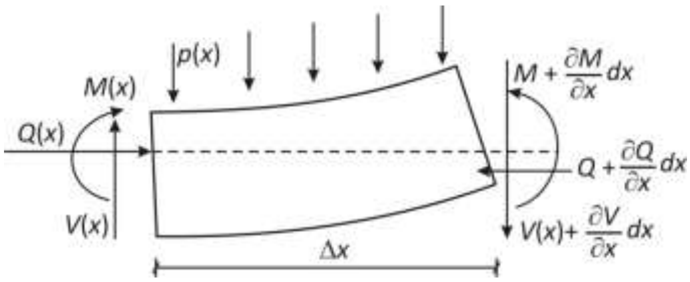

Fig-1: Free body diagram of an infinitesimal column segment

The ordinary differential equation of equilibrium of elastic column buckling is formulated using principles that are similar to those for flexural behaviour of beams, except that the effects of axial compressive forces are included in the consideration of forces for equilibrium. Considering equilibrium of forces in the horizontal direction,

$$\sum F_x = Q(x) + \frac{\partial Q(x)}{\partial x}\Delta x - Q(x) = 0 \quad (1)$$

$$\frac{\partial Q(x)}{\partial x}\Delta x = 0 \quad (2)$$

or $$\frac{\partial Q(x)}{\partial x} = 0 \quad (3)$$

$Q(x)$ is a constant.

In the Figure-1, $M(x)$ is the bending moment at the left end, $M(x) + dM(x) = M(x) + \frac{\partial M(x)}{\partial x}\Delta x$ is the bending moment at the right end Δx apart from the left end. Similarly, $V(x)$ is the shear force at the left end,

$V(x)+\frac{\partial V(x)}{\partial x}\Delta x$ is the shear force at the right end. $Q(x)$ is the axial force at the left end. $Q(x)+\frac{\partial Q}{\partial x}\Delta x$ is the axial force at the right end. $p(x)$ is the distribution of transverse load on the column Δw is transverse deflection.

Considering equilibrium of forces in the vertical direction,

$$V(x)+\frac{\partial V(x)}{\partial x}\Delta x - V(x) + p(x)\Delta x = 0 \qquad (4)$$

$$\frac{\partial V(x)}{\partial x}\Delta x = -p(x)\Delta x \qquad (5)$$

$$\frac{\partial V(x)}{\partial x} = -p(x) \qquad (6)$$

Considering equilibrium of forces in rotation, about point O, at the right end of the column segment,

$$M(x)+\frac{\partial M(x)}{\partial x}\Delta x - M(x) + p(x)\Delta x\frac{\Delta x}{2} - V(x)\Delta x - Q(x)\Delta w = 0 \qquad (7)$$

$$\frac{\partial M(x)}{\partial x}\Delta x + p(x)\frac{(\Delta x)^2}{2} - V(x)\Delta x - Q(x)\Delta w = 0 \qquad (8)$$

$$\frac{\partial M(x)}{\partial x} - Q(x)\frac{\Delta w}{\Delta x} = V(x) - p(x)\frac{\Delta x}{2} \qquad (9)$$

In the limit as $\Delta x \to 0$,

$$\frac{dM(x)}{dx} - Q(x)\frac{dw(x)}{dx} = V(x) \qquad (10)$$

By differentiation of Equation (10) with respect to x, we have:

$$\frac{d}{dx}\left(\frac{dM(x)}{dx} - Q(x)\frac{dw(x)}{dx}\right) = \frac{dV(x)}{dx} = -p(x) \qquad (11)$$

Simplifying,

$$\frac{d^2M(x)}{dx^2} - Q(x)\frac{d^2w(x)}{dx^2} = -p(x) \qquad (12)$$

The bending moment – displacement relationship for Euler – Bernoulli beam theory is:

$$M(x) = -EI\frac{d^2w(x)}{dx^2} \qquad (13)$$

Then, Equation (12) becomes, on using Equation (13)

$$\frac{d^2}{dx^2}\left(-EI\frac{d^2w(x)}{dx^2}\right) - Q\frac{d^2w(x)}{dx^2} = -p(x) \qquad (14)$$

For prismatic cross-sections,

$$-EI\frac{d^4w(x)}{dx^4} - Q\frac{d^2w(x)}{dx^2} = -p(x) \quad (15)$$

or, $$EI\frac{d^4w(x)}{dx^4} + Q(x)\frac{d^2w(x)}{dx^2} = p(x) \quad (16)$$

Where,

$0 \leq x \leq l.$

Equation (16) which is a fourth order linear ordinary differential equation in terms of $w(x)$ is the governing equation for elastic buckling of Euler columns.

METHODOLOGY

A one term least squares weighted residual solution to the elastic buckling problem is found by assuming a one parameter displacement field as:

$$w(x) = a_1\varphi_1(x) \quad (17)$$

Where a_1 is an unknown generalised displacement parameter and $\varphi_1(x)$ is the displacement shape function chosen to satisfy the boundary conditions. The least squares weighted residual integral then becomes

$$\int_0^l \frac{\partial w(x)}{\partial a_1}(EIw^{iv}(x) + Qw''(x))\,dx = 0 \quad (18)$$

$$\int_0^l \varphi_1(x)\left(EI(a_1\varphi_1(x))^{iv} + Q(a_1\varphi_1(x))''\right)dx = 0 \quad (19)$$

$$\int_0^l \varphi_1(x)a_1\left(\varphi_1^{iv}(x) + \frac{Q}{EI}\varphi_1''(x)\right)dx = 0 \quad (20)$$

$$a_1\left\{\int_0^l \varphi_1^{iv}(x)\varphi_1(x)\,dx + \frac{Q}{EI}\int_0^l \varphi_1''(x)\varphi_1(x)\,dx\right\} = 0 \quad (21)$$

$$a_1(k_{11} + \beta k_{11g}) = 0 \quad (22)$$

$$k_{11} = \int_0^l \varphi_1^{iv}(x)\varphi_1(x)\,dx \quad (23)$$

$$k_{11g} = \int_0^l \varphi_1''(x)\varphi_1(x)\,dx \quad (24)$$

$$\beta = \frac{Q}{EI} \quad (25)$$

β is the buckling load factor. For non-trivial solutions $a_1 \neq 0$, and the characteristic buckling equation for the one-parameter least squares weighted residual solution becomes the algebraic eigenvalue – eigenvector problem.

$$\left|k_{11} + \lambda k_{11g}\right| = 0 \tag{26}$$

Expanding and solving,

$$\lambda = \frac{-k_{11}}{k_{11g}} \tag{27}$$

The critical buckling load can thus be determined.
For two parameter displacement fields,

$$w(x) = a_1\varphi_1(x) + a_2\varphi_2(x) \tag{28}$$

Where a_1 and a_2 are the two unknown generalised parameters used to define the displacement field, and $\varphi_1(x)$, $\varphi_2(x)$ are the displacement buckling shape functions chosen to satisfy the boundary conditions.

The least squares weighted residual integrals become the system of two equations:

$$\int_0^l \frac{\partial w(x)}{\partial a_1}\left[(a_1\varphi_1(x) + a_2\varphi_2(x))^{iv} + \frac{Q}{EI}(a_1\varphi_1(x) + a_2\varphi_2(x))''\right]dx = 0 \tag{29}$$

$$\int_0^l \frac{\partial w(x)}{\partial a_2}\left[(a_1\varphi_1(x) + a_2\varphi_2(x))^{iv} + \frac{Q}{EI}(a_1\varphi_1(x) + a_2\varphi_2(x))''\right]dx = 0 \tag{30}$$

Expanding,

$$a_1\left(\int_0^l \varphi_1^{iv}(x)\varphi_1(x)\,dx + \frac{Q}{EI}\int_0^l \varphi_1''(x)\varphi_1(x)\,dx\right) + a_2\left(\int_0^l \varphi_2^{iv}(x)\varphi_1(x)\,dx + \frac{Q}{EI}\int_0^l \varphi_2''(x)\varphi_1(x)\,dx\right) = 0$$
…(31)

$$a_1\left(\int_0^l \varphi_1^{iv}(x)\varphi_2(x)\,dx + \frac{Q}{EI}\int_0^l \varphi_1''(x)\varphi_2(x)\,dx\right) + a_2\left(\int_0^l \varphi_2^{iv}(x)\varphi_2(x)\,dx + \frac{Q}{EI}\int_0^l \varphi_2''(x)\varphi_2(x)\,dx\right) = 0$$
…(32)

In matrix format,

$$\begin{pmatrix} k_{11} + \beta k_{11g} & k_{12} + \beta k_{12g} \\ k_{21} + \beta k_{21g} & k_{22} + \beta k_{22g} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \tag{33}$$

Where

$$k_{12} = \int_0^l \varphi_2^{iv}(x)\varphi_1(x)\,dx \tag{34}$$

$$k_{12}g = \int_0^l \varphi_2''(x)\varphi_1(x)\,dx \tag{35}$$

$$k_{21} = \int_0^l \varphi_1^{iv}(x)\varphi_2(x)\,dx \quad (36)$$

$$k_{21}g = \int_0^l \varphi_1''(x)\varphi_2(x)\,dx \quad (37)$$

$$k_{22} = \int_0^l \varphi_2^{iv}(x)\varphi_2(x)\,dx \quad (38)$$

$$k_{22}g = \int_0^l \varphi_2''(x)\varphi_2(x)\,dx \quad (39)$$

For non-trivial solutions,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \neq 0$$

The buckling equation becomes:

$$\begin{vmatrix} k_{11} + \beta k_{11g} & k_{12} + \beta k_{12g} \\ k_{21} + \beta k_{21g} & k_{22} + \beta k_{22g} \end{vmatrix} = 0 \quad (40)$$

Expanding, the characteristic buckling equation is:

$$(k_{11} + \beta k_{11g})(k_{22} + \beta k_{22g}) - (k_{12} + \beta k_{12g})(k_{21} + \beta k_{21g}) = 0 \quad (41)$$

The characteristic buckling equation is a quadratic equation with two roots from which the buckling loads are determined.

RESULTS

Elastic buckling problem considered

The Euler column considered was fixed at one end, considered x =0, end pinned at the other end, $x = l$ as shown in Figure-2.

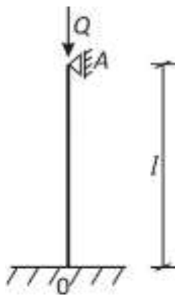

Fig-2:

The elastic buckling problem is given on the domain $0 \le x \le l$ by Equation (16). The boundary conditions are

$$w(x = 0) = \theta(x = 0) = w'(x = 0) = 0 \quad (42)$$

$$w(x = l) = w''(x = l) = 0 \quad (43)$$

A suitable displacement buckling shape function that satisfies the boundary conditions at $x = 0$, and $x = l$ in a one-parameter least squares weighted residual formulation is:

$$w(x) = a_1\varphi_1(x) = a_1\left(\left(\frac{x}{l}\right)^4 - 2.5\left(\frac{x}{l}\right)^3 + 1.5\left(\frac{x}{l}\right)^2\right) \quad (44)$$

For a two parameter buckling modal shape function, the second buckling modal shape function that satisfies the boundary conditions is:

$$\varphi_2(x) = \left(\frac{x}{l}\right)^5 - \frac{7}{3}\left(\frac{x}{l}\right)^4 + \frac{4}{3}\left(\frac{x}{l}\right)^3 \quad (45)$$

One parameter least squares weighted residual solution

By differentiation,

$$\varphi_1^{iv}(x) = \frac{24}{l^3} \quad (46)$$

$$\varphi_1''(x) = \frac{12x^2}{l^4} - \frac{15x}{l^3} + \frac{3}{l^2} \quad (47)$$

$$k_{11} = \int_0^l \varphi_1^{iv}(x)\varphi_1(x)\,dx = \int_0^l \frac{24}{l^4}\left(\left(\frac{x}{l}\right)^4 - 2.5\left(\frac{x}{l}\right)^3 + 1.5\left(\frac{x}{l}\right)^2\right)dx = \frac{1.8}{l^3} \quad (48)$$

$$k_{11g} = \int_0^l \varphi_1''(x)\varphi_1(x)dx = \int_0^l \left(\frac{12x^2}{l^4} - \frac{15x}{l^3} + \frac{3}{l^2}\right)\left(\left(\frac{x}{l}\right)^4 - 2.5\left(\frac{x}{l}\right)^3 + 1.5\left(\frac{x}{l}\right)^2\right)dx = -\frac{0.085714286}{l} \quad (49)$$

Then,

$$a_1\left(\frac{1.8}{l^3} - \frac{0.085714286}{l}\frac{Q}{EI}\right) = 0 \quad (50)$$

The characteristic buckling equation is the homogeneous equation:

$$\frac{1.8}{l^3} - \frac{Q}{EI}\frac{0.085714286}{l} = 0 \quad (51)$$

Solving,

$$Q_{cr} = 21\frac{EI}{l^2} \quad (52)$$

Two parameter least squares weighted residual solution

By differentiation,

$$\varphi_2''(x) = \frac{20x^3}{l^5} - \frac{28x^2}{l^4} + \frac{8x}{l^3} \quad (53)$$

$$\varphi_2''(x) = \frac{120x}{l^5} - \frac{56}{l^4} \quad (54)$$

By integration,

$$k_{12} = \int_0^l \left(\frac{120x}{l^5} - \frac{56}{l^4} \right) \left(\left(\frac{x}{l}\right)^4 - 2.5\left(\frac{x}{l}\right)^3 + 1.5\left(\frac{x}{l}\right)^2 \right) dx = \frac{0.8}{l^3} \quad (55)$$

$$k_{12g} = \int_0^l \varphi_2''(x)\varphi_1(x)\,dx = \int_0^l \left(\frac{20x^3}{l^5} - \frac{28x^2}{l^4} + \frac{8x}{l^3} \right) \left(\left(\frac{x}{l}\right)^4 - 2.5\left(\frac{x}{l}\right)^3 + 1.5\left(\frac{x}{l}\right)^2 \right) dx$$

$$= \frac{-0.042571429}{l} \quad (56)$$

$$k_{21} = \int_0^l \frac{24}{l^4} \left(\left(\frac{x}{l}\right)^5 - \frac{7}{3}\left(\frac{x}{l}\right)^4 + \frac{4}{3}\left(\frac{x}{l}\right)^3 \right) dx = \frac{0.8}{l^3} \quad (57)$$

$$k_{21g} = \int_0^l \left(\frac{12x^2}{l^4} - \frac{15x}{l^3} + \frac{3}{l^2} \right) \left(\left(\frac{x}{l}\right)^5 - \frac{7}{3}\left(\frac{x}{l}\right)^4 + \frac{4}{3}\left(\frac{x}{l}\right)^3 \right) dx = \frac{-0.042857142}{l} \quad (58)$$

$$k_{22} = \int_0^l \left(\frac{120x}{l^5} - \frac{56}{l^4} \right) \left(\left(\frac{x}{l}\right)^5 - \frac{7}{3}\left(\frac{x}{l}\right)^4 + \frac{4}{3}\left(\frac{x}{l}\right)^3 \right) dx = \frac{0.609523734}{l^3} \quad (59)$$

$$k_{22g} = \int_0^l \left(\frac{20x^3}{l^5} - \frac{28x^2}{l^4} + \frac{8x}{l^3} \right) \left(\left(\frac{x}{l}\right)^5 - \frac{7}{3}\left(\frac{x}{l}\right)^4 + \frac{4}{3}\left(\frac{x}{l}\right)^3 \right) dx = \frac{-0.025396824}{l} \quad (60)$$

Then, the least squares weighted residual equations become:

$$a_1\left(\frac{1.8}{l^3} - \beta\frac{0.085714286}{l} \right) + a_2\left(\frac{0.8}{l^3} - \beta\frac{0.0428571429}{l} \right) = 0 \quad (61)$$

$$a_1\left(\frac{0.8}{l^3} - \beta\frac{0.042857142}{l} \right) + a_2\left(\frac{0.609523734}{l^3} - \frac{0.025396824}{l}\beta \right) = 0 \quad (62)$$

In matrix form,

$$\begin{pmatrix} (1.80 - 0.085714286\beta l^2) & (0.80 - 0.0428571429\beta l^2) \\ (0.80 - 0.042857142\beta l^2) & (0.609523734 - 0.025396824\beta l^2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (63)$$

For non-trivial solutions,

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \neq 0$$

The characteristic buckling equation then becomes

$$\begin{vmatrix} (1.80 - 0.085714\lambda) & (0.80 - 0.042857\lambda) \\ (0.80 - 0.042857\lambda) & (0.609524 - 0.0253968\lambda) \end{vmatrix} = 0 \quad (64)$$

Where

$$\lambda = \beta l^2 = \frac{Ql^2}{EI} \quad (65)$$

Thus, expanding,

$$(1.80 - 0.085714\lambda)(0.609524 - 0.0253968\lambda) - (0.80 - 0.042857\lambda)^2 = 0 \quad (66)$$

Simplifying,

$$\lambda^2 - 87.13621\lambda + 1358.92 = 0 \quad (67)$$

Solving,

$$\lambda^2 = 20.34614 = \frac{Ql^2}{EI} \quad (68)$$

Then

$$Q_{cr} = 20.34614\frac{EI}{l^2} \quad (69)$$

The exact solution for critical elastic buckling load ($Q_{cr\,\text{exact}}$) of Euler column with fixed pinned ends is

$$Q_{cr\,\text{exact}} = 20.1907\frac{EI}{l^2} \quad (70)$$

DISCUSSION

The elastic buckling problem of Euler columns under axial compressive load Q when the ends are fixed at $x = 0$, and pinned at $x = l$ has been solved in this work using the least squares weighted residual method. One and two parameter buckling displacement shape functions constructed as polynomial splines that satisfy all the boundary conditions were used to express the boundary value problem as Equation (50) for the one – parameter buckling shape function case, and as Equations (61) and (68) for the two parameter buckling shape function. The critical buckling load for the one parameter case was found as Equation (52) by solving the characteristic buckling equation expressed as Equation (51).

Similarly, the two parameter buckling shape function resulted in the algebraic eigenvalue – eigenvector problem represented by Equation (63). The corresponding characteristic buckling equation was found as Equation (64). Equation (64) yielded on expansion, a quadratic equation (Equation (67)) in terms of λ which was solved to obtain the two roots with the root that gave the lowest value of Q used to obtain the critical buckling load.

Comparison of the critical buckling load results of one parameter and two parameter buckling shape functions, with the exact critical buckling load given as Equation (70) shows the one parameter solution has a relative difference of 41% while the two parameter least squares weighted residual solution has a relative difference of 0.77%. This work illustrates the effectiveness of the least squares weighted residual method in the elastic buckling analysis of Euler columns with one end fixed and the other end pinned for the case of axial compressive load, Q.

CONCLUSIONS

The conclusions of the study are as follows:

- The least squares weighted residual method reduces the solution of the boundary value problem of elastic buckling of Euler columns represented by a fourth order ordinary differential equation to an algebraic eigenvalue eigenvector problem.
- The least squares weighted residual method gives reasonably accurate solution to the elastic buckling problem of Euler column with fixed pinned end even with a one parameter buckling shape function provided all the boundary conditions are satisfied.
- The accuracy of the method improves with the increase in the number of modal buckling shape functions, provided all the buckling shape functions satisfy all the boundary conditions.
- The problem is simplified to the computation of elastic stiffness and geometric stiffness terms in the matrix representing the buckling problem in the formulation.

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