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Review Article

Application of Weak-Form Variational Method in Nonlinear Free Vibration Analysis of Highly Clamped Rectangular Plates

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Abstract

This paper utilizes weak-form variational principle to determine the nonlinear natural frequencies of highly clamped rectangular isotropic plates. The energy functional was formulated using weak-form variational statement on the integral function of the plate problem. The displacement functions were developed based on static deflections of orthogonal beam network. The algebraic expression for stress function was determined using direct integration process on compatibility equation. The amplitude of deflection which influences the geometric nonlinearity of the plate was determined using integration process on energy functional based on static equilibrium equations. The stiffness and mass matrices were developed from the expressions of energy functional based on dynamic equilibrium equations. The nonlinear natural frequencies were numerically computed. The validation of the results using the results from previous work found in literature shows satisfactory convergence, with an error of 0.034%. Conclusively, weak-form variational principle gives satisfactory approximation to nonlinear vibration analysis of rectangular isotropic plates.
Keywords: Algebraic polynomial, Amplitude of deflection, Energy functional, Nonlinear natural frequency, Rectangular plate, Weak-form, Variational principle.

INTRODUCTION

A highly clamped rectangular plate in the context of this paper refers to any rectangular plate with at least three of its edges rigidly fixed. Engineering analysis of clamped rectangular plates is problematic; and there is no known closed-form solution to clamped rectangular plates. Consequently, approximation techniques are usually employed in the solution of rectangular clamped plates.

Practically, most of the engineering applications of rectangular plates are in the form of thin-walled structures. Furtherance to their thinness, two main precisions of approximation are commonly adopted in the rectangular thin plate's analysis, namely the linear mathematical model and the nonlinear mathematical model. However, the linear method of approximation is just only but first order approximation [1]. In nonlinear method of approximation, the deflection is assumed to be appreciable compared to the plate thickness; and the nonlinear method of approximation yields solution closer to the exact solution to the plate problem. On the other hand, the resulting nonlinear differential equations from nonlinear method of analysis are tedious to tackle. In particular, the determination of a series of eigen-frequencies for large amplitude of vibrations of rectangular plates based on analytical technique approach poses highly technical challenges; and most often only the first mode frequency can be determined even with the advent of high speed computers [2]. Consequently, a host of investigators on large amplitude of vibrations employ different approximation techniques to tackle the problems of nonlinear vibration analysis of rectangular plates.

Large amplitudes of vibrations of rectangular plates have for sometimes now posed interesting and challenging research fields. Different approximation techniques have been employed to investigate into large amplitude vibrations of rectangular plates [3,4] used finite element formulation to investigate large amplitude vibrations of rectangular plates. The Hierarchical finite element and Continuation methods were employed to study nonlinear vibration of plates [5]. The Spline strip method was used by [6]. Energy principle based on Hamilton's principle was used by

[7]. An Optimal Artificial Neural Network was used to predict large amplitude vibrations of laminated composite rectangular plates based on first order shear deformation theory [8]. The Galerkin-iterative method was applied to investigate the nonlinear vibrations of rectangular plates by [9]. Furthermore, [10] employed the Harmonic Balance Computation to study the nonlinear frequency response of a thin plate. [11] studied the small-scale effect on nonlinear free vibration of isotropic thin nano-plate by using a combination of nonlocal elasticity plate theory and the Von Karman geometric model [2, 12, 13] published books that deal with nonlinear plate analysis.

In energy approximation techniques, the uses of Hamilton' principle, direct variational principle in Ritz method and residual energy method in Galerkin approach have been common to many investigators. The unabated applications of these techniques have demonstrated questionable monotonous characteristics to some classes of readership. Furthermore, the application of trigonometric functions in the form of Fourier series has been used by most of the researchers as approximation solutions; despite to the fact that trigonometric functions cannot provide satisfactory solutions to all the boundary conditions of rectangular plates.

The objectives of this study shall include the application of weak-form variational method in formulating energy functional for the rectangular plate problem. Secondly this study is poised to develop algebraic polynomial functions based on orthogonal beam network displacement functions as shape functions. Finally, this study is expected to determine the nonlinear natural frequencies beyond the first mode frequency of the rectangular plates.

THEORETICAL FORMULATION

Formulation of Energy Functional

The formulation of energy functional is based on the application of weak-form variational principle on integral function of Von Karman's thin plate differential equations. The analogous Von Karman's thin plate differential equations of motion are as expressed in Eqs. (1) and (2).

$$\nabla^4 w = \frac{1}{D}\left[q + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \rho t \ddot{w}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\nabla^4 F = Et\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where

w = displacement functions
F = Airy's stress functions
t = plate thickness
ρ = mass density of plate materials
q = uniformly applied lateral load
$\ddot{w}$ = second derivative of w with respect to time T
D = flexural rigidity of the plate, defined as given in Eq. (3)

$$D = \frac{Et^3}{12(1-\mu^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Where

μ = Poisson's ratio

The variational statement is applied on the integral functions of Eq. (1) as expressed in Eq. (4).

$$0 = \int_0^a \int_0^b V(x,y)\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - K\left(q + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \rho t \ddot{w}\right)\right] dxdy \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Where

$V(x, y)$ = the weighting function

$$K = \frac{1}{D} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Then integration by parts is performed on Eq. (4) to trade differentiation from w and F to V(x, y), which on simplification yields Eq. (6).

$$0 = \int_0^a \int_0^b \left[\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 V}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial y^2} \pm 2KF\frac{\partial^2 V}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} \pm KF\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial x^2} \pm KF\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - KVq + KV\rho t \ddot{w}\right] dxdy + \int_0^a \left\{\left[2\frac{\partial V}{\partial x}(M_{xy}) - V\left(\frac{\partial M_y}{\partial y}\right) + \frac{\partial V}{\partial y}(M_y) + 2KF\frac{\partial V}{\partial x}(M_{xy}) + KV\frac{\partial F}{\partial y}(M_x) + 2KVF\left(\frac{\partial M_{xy}}{\partial x}\right) -\right.\right.$$

$$KF\frac{\partial V}{\partial y}(M_x)\Big]_0^b\Big\}dx + \int_0^b\Big\{\Big[-V\left(\frac{\partial M_x}{\partial x}\right)+\frac{\partial V}{\partial x}(M_x)-2V\left(\frac{\partial M_{xy}}{\partial y}\right)-KF\frac{\partial V}{\partial y}(M_{xy})-KVF\left(\frac{\partial M_{xy}}{\partial y}\right)-2KV\frac{\partial F}{\partial y}(M_{xy})- KF\frac{\partial V}{\partial x}(M_y)+KV\frac{\partial F}{\partial x}(M_y)\Big]_0^a\Big\}dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Where

$V = V(x, y)$ …………………….. (7)

In Eq. (6), the last two integrands that are in the curl brackets are the natural boundary conditions; and each of them is identically satisfied at the boundaries; and as such they are dropped in the further discussion. The integrand in the double integration sign of Eq. (6) is the functional of the plate problem.

By assuming that the plate vibrates harmonically and performs sinusoidal time response, the inertia term is simplified, and the expression for the functional is as expressed in Eq. (8).

$$0=\int_0^a\int_0^b\Big[\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial x^2}+2\frac{\partial^2 V}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial y^2}\pm 2KF\frac{\partial^2 V}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\pm KF\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\pm KF\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-KVq- KV\rho t\omega^2 w\Big]\,dxdy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Furthermore, for a free vibration analysis, the lateral load q is disregarded so that Eq. (8) is simplified as expressed in Eq. (9).

$$0=\int_0^a\int_0^b\Big[\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial x^2}+2\frac{\partial^2 V}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial y^2}\pm 2KF\frac{\partial^2 V}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\pm KF\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\pm KF\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\omega^2 MKVw\Big]\,dxdy \quad \ldots\ldots\ldots\ldots\ldots (9)$$

Where

$M = \rho * t$ …………….. (10)

The General Boundary Conditions

In dealing with geometrically nonlinear analysis of rectangular plates, the out-of-plane and in-plane boundary conditions are jointly considered; and these are stated as follows [12, 14, 15]:

Rigidly Clamped at the four Edges (CCCC)

$$u=v=w=w_o=\frac{\partial w}{\partial x}=\frac{\partial w_o}{\partial x}=0\ at\ x=0,a \ldots\ldots\ldots\ldots (11)$$

$$u=v=w=w_o=\frac{\partial w}{\partial y}=\frac{\partial w_o}{\partial y}=0\ at\ y=0,b \ldots\ldots\ldots\ldots (12)$$

Clamped – Clamped – Clamped – Simply Supported (CCCS)

$$u=v=w=w_o=\frac{\partial w}{\partial x}=\frac{\partial w_o}{\partial x}=0\ at\ x=0,a \ldots\ldots\ldots\ldots (13)$$

$$u=v=w=w_o=\frac{\partial w}{\partial y}=\frac{\partial w_o}{\partial y}=0\ at\ y=0 \ldots\ldots\ldots\ldots (14)$$

$$u=v=w=w_o=N_{yx}=M_y=M_{yx}=\frac{\partial^2 w_o}{\partial y^2}=0\ at\ y=b \ldots\ldots\ldots (15)$$

Clamped – Clamped – Clamped – Free Edge (CCCF)

$$u=v=w=w_o=\frac{\partial w}{\partial x}=\frac{\partial w_o}{\partial x}=0\ at\ x=0,a \ldots\ldots\ldots\ldots (16)$$

$$u=v=w=w_o=\frac{\partial w}{\partial y}=\frac{\partial w_o}{\partial y}=0\ at\ y=0 \ldots\ldots\ldots\ldots (17)$$

$$M_y=Q_y=N_y=N_{yx}=0\ at\ y=b \ldots\ldots\ldots\ldots (18)$$

Determination of Specially Energy Functional

Eq. (9) gives generic energy functional based on the integral function of the plate problem; but to determine the specially energy functional with respect to any set of rectangular plate boundary conditions, the general boundary conditions of Eqs. (11) through (18) are appropriately applied. From the general boundary conditions, the Specially Energy Functional (SEF) for CCCC and CCCS rectangular plates is given as expressed in Eq. (19).

$$0 = \int_0^a \int_0^b \left[\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 V}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial y^2} + 2KF\frac{\partial^2 V}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + KF\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + KF\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \omega^2 MKVw\right] dxdy \quad \ldots\ldots\ldots\ldots\ldots (19)$$

For CCCF rectangular plate, the SEF is given as expressed in Eq. (20).

$$0 = \int_0^a \int_0^b \left[\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 V}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial y^2} + 2KF\frac{\partial^2 V}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + KF\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - \omega^2 MKVw\right] dxdy \quad \ldots\ldots\ldots\ldots\ldots (20)$$

Development of Displacement Function

In this study, the displacement functions are sought in the form of algebraic polynomial displacement functions. In reality, each vibration mode shape of a rectangular plate consists of a product of two perpendicular motions in x- and y-coordinates respectively. Consequently, a rectangular plate can be idealized to consist of a series of imaginary orthogonal beam network as shown in Fig-1.

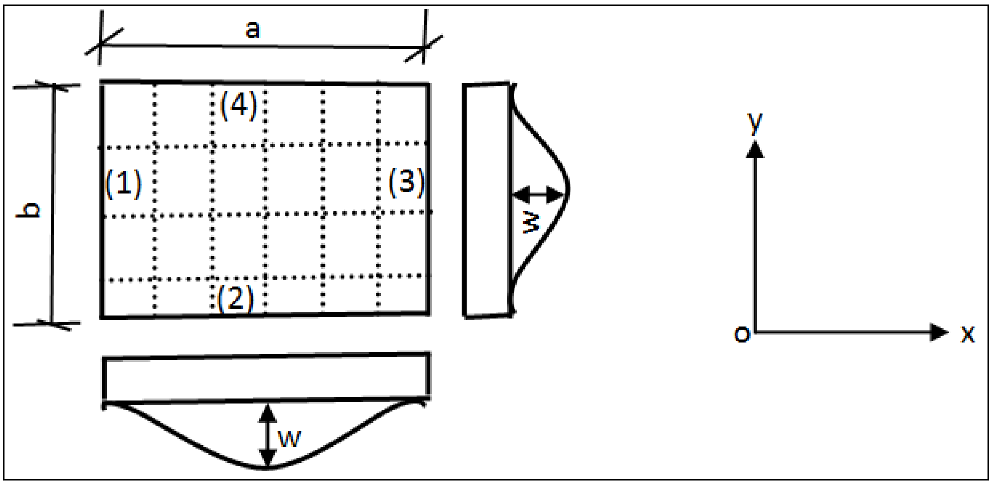

Fig-1: Showing a Series of Orthogonal Beam Network and Deflection Configurations

It is assumed that each beam element running along any coordinate direction is a good representative of other series of beam elements that run along the same direction. Let *w(x)* and *w(y)* denote the algebraic polynomial displacement functions in x- and y-directions respectively. Thus:

$$w(x) = \sum_{i=0}^{k} c_i x^i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

$$w(y) = \sum_{j=0}^{k} d_j y^j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Where c_i and d_j are undetermined coefficients.

Mathematically it has been established that a five-term polynomials provides necessary and sufficient satisfaction for completeness as shape functions. Therefore the value of k in the summation is taken to be four. The out-of-plane boundary conditions for clamped edge, simply supported edge and free edge are as given in Eqs. (23), (24) and (25) respectively.

$$w(\cdot,\cdot) = w'(\cdot,\cdot) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

$$w(\cdot,\cdot) = w''(\cdot,\cdot) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

$$w'(\cdot,\cdot) = w''(\cdot,\cdot) = w'''(\cdot,\cdot) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

Where $w(\cdot,\cdot)$ represents deflection function, $w'(\cdot,\cdot)$, $w''(\cdot,\cdot)$ and $w'''(\cdot,\cdot)$ represent first, second and third derivatives with respect to the coordinates respectively.

Suppose the edges (1) and (3) of Fig-1 are clamped, and by invoking Eq. (21), the five term polynomial expression is given as in Eq. (26).

$$w(x) = c_o + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

By applying the boundary conditions of Eq. (23) and subsequently carrying out simplification yields Eq. (27).

$$w(x) = c_4 * a^4 \left[\left(\frac{x}{a}\right)^2 - 2\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)^4\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

However, to account for series of beam mode shapes running along y-direction, Eq. (27) is modified as shown in Eq. (28).

$$w(x) = c_4 * a^4 \left[\left(\frac{x}{a}\right)^{m+1} - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

Similarly suppose the edges (2) and (4) of Fig. 1 are clamped, then the polynomial expression is:

$$w(y) = d_o + d_1 y + d_2 y^2 + d_3 y^3 + d_4 y^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

By applying the required boundary conditions and simplifying, yields Eq. (30).

$$w(y) = d_4 * b^4 \left[\left(\frac{y}{b}\right)^2 - 2\left(\frac{y}{b}\right)^3 + \left(\frac{y}{b}\right)^4\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

Also, to account for series of beam mode shapes running along x-direction, Eq. (30) is modified as shown in Eq. (31).

$$w(y) = d_4 * b^4 \left[\left(\frac{y}{b}\right)^{n+1} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

The rectangular plate static mode shape is the product of Eq. (28) and Eq. (31) as expressed in Eq. (32).

$$w(x,y) = A_{mn} \left[\left(\frac{x}{a}\right)^{m+1} - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3}\right]\left[\left(\frac{y}{b}\right)^{n+1} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

Where

$$A_{mn} = (c_4 * a^4)(d_o * b^4) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

The parameter A_{mn} is the amplitude of deflection function. Thus Eq. (32) is the static algebraic polynomial displacement function for CCCC rectangular plate. The same procedure is used to develop the static algebraic polynomial mode shapes for CCCS and CCCF rectangular plates; and they are as expressed in Eqs. (34) and (35) respectively.

$$w(x,y) = A_{mn} \left[\left(\frac{x}{a}\right)^{m+1} - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3}\right]\left[\frac{3}{2}\left(\frac{y}{b}\right)^{n+1} - \frac{5}{2}\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

$$w(x,y) = A_{mn} \left[\left(\frac{x}{a}\right)^{m+1} - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3}\right]\left[4\left(\frac{y}{b}\right)^{n+1} - 4\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

The Stress Function

One of the unknown parameters in the expression for the energy functional is the Airy's stress function F. In this work, direct integration process on compatibility equation of Eq. (2) is used to determine the expression for stress function. Thus, a repeated integration process on the right hand side of Eq. (2) is performed four times with respect to x-coordinate; and then four times with respect to y-coordinate as expressed in Eq. (36).

$$F_{mn} = \int_x^{\cdot}\int_x^{\cdot}\int_x^{\cdot}\int_x^{\cdot}\int_y^{\cdot}\int_y^{\cdot}\int_y^{\cdot}\int_y^{\cdot}\left[\left(\frac{\partial^2 w(mn)}{\partial x \partial y}\right)^2 - \frac{\partial^2 w(mn)}{\partial x^2}\frac{\partial^2 w(mn)}{\partial y^2}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

Where $\int_x^{\cdot} = \int(\cdot)dx$ and $\int_y^{\cdot} = \int(\cdot)dy$; and for *m*, *n* = 1, 2

In this study, the following designations for describing mode shapes are adopted: for *m* = *n* = 1, then w(1,1) denotes the first mode shape; for *m* = 1 and *n* = 2, then w(1,2) denotes the second mode shape; for *m* = 2 and *n* = 1, then w(2,1) denotes the third mode shape; and for *m* = 2 and *n* = 2, then w(2,2) denotes the fourth mode shape. For instance, at mode 1, the mode shape with respect to CCCC rectangular plate is as given in Eq. (37).

$$w(1,1) = A_{11} \left[\left(\frac{x}{a}\right)^2 - 2\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)^4\right]\left[\left(\frac{y}{b}\right)^2 - 2\left(\frac{y}{b}\right)^3 + \left(\frac{y}{b}\right)^4\right] \ldots\ldots\ldots\ldots\ldots\ldots (37)$$

Therefore the corresponding algebraic stress function F_{11} with respect to CCCC rectangular plate evaluated using Eq. (36) is as expressed in Eq. (38).

$$F_{11} = \frac{Qx^6y^6A_{11}^2}{6350400a^8b^8}[-96a^3x(14b^4 - 30b^3y + 30b^2y^2 - 15by^3 + 3y^4) + 14a^4(42b^4 - 96b^3y + 99b^2y^2 - 50by^3 + 10y^4) - 10ax^3(70b^4 - 144b^3y + 141b^2y^2 - 70by^3 + 14y^4) + 2x^4(70b^4 - 144b^3y + 141b^2y^2 - 70by^3 + 14y^4) + 3a^2x^2(462b^4 - 960b^3y + 945b^2y^2 - 470by^3 + 94y^4)] \quad \ldots\ldots (38)$$

Where

$$Q = E * t \ldots\ldots (39)$$

At any specified mode, the corresponding stress function is determined by substituting the corresponding mode shape into compatibility equation of Eq. (2); and then integrated using Eq. (36).

Determination of Amplitude of Deflection

Under large deflection scenario, the analysis of rectangular plate is shifted from rigid to flexible plate analysis. The rigidity parameters associated with flexible plates are variable, and highly depend on the deflection as well as the lateral loading [2]. Practically, the frequency of natural vibration is a function of the rigidity parameter of the plate; and under large deflection, the frequency depends on how much the system deviates from equilibrium position (amplitude of deflection).

The amplitude of deflection A_{mn} is unknown parameter associated with the mode shape; and to determine the amplitude of deflection, the expression for mode shape is appropriately substituted into the energy functional equation. The inertia parameter in the energy equation is replaced with the lateral load parameter.

In this work, the mode shapes to be considered for any given set of plate boundary conditions are *w(1,1)*, *w(1,2)*, *w(2,1)* and *w(2,2)*. Therefore, the amplitude of deflection to be determined are A_{11}, A_{12}, A_{21} and A_{22} respectively. It is assumed that the problem system is self-adjoint. Consequently, the weighting function *V* in the energy functional is interchanged with the deflection function *w* without loss in generality. For instance, by letting V = w(1,1), the integral expression for evaluating the amplitude of deflection A_{11} for CCCC rectangular plate is as shown in Eq. (40).

$$\int_0^a \int_0^b \left[\left(\frac{\partial^2 w(1,1)}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w(1,1)}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w(1,1)}{\partial y^2}\right)^2 + 2KF\left(\frac{\partial^2 w(1,1)}{\partial x \partial y}\right)^2 + 2KF\left(\frac{\partial^2 w(1,1)}{\partial x^2}\right)\left(\frac{\partial^2 w(1,1)}{\partial y^2}\right)\right] dxdy = \int_0^a \int_0^b Kqw(1,1)dxdy \ldots\ldots (40)$$

By carrying out double integration process and simplification on Eq. (40), an expression that simulates the Duffing's type of equation is obtained as shown in Eq. (41).

$$1.5 * 10^{-8}a^4\beta^4KQA_{11}^3 + [1.142857(\beta^4 + 1) + 0.653061\beta^2]A_{11} - a^4\beta^4Kq = 0 \ldots\ldots (41)$$

Where $= \frac{b}{a}$. The real root of Eq. (41) in terms of A_{11} is given as shown in Eq. (42).

$$A_{11} = \frac{-1.25992\xi_1}{\left[Z_1 + \sqrt{Z_2 + Z_3\xi_1^3}\right]^{\frac{1}{3}}} + \zeta_1\left\{1.76378 * 10^7\left[Z_1 + \sqrt{Z_2 + Z_3\xi_1^3}\right]^{\frac{1}{3}}\right\} \ldots\ldots (42)$$

Where

$$\xi_1 = (1.14286 + 0.653061\beta^2 + 1.14286\beta^4) \ldots\ldots (43)$$

$$Z_1 = 6.075 * 10^{-15}a^{12}K^3qQ^2\beta^{12} \ldots\ldots (44)$$

$$Z_2 = 3.69056 * 10^{-29}a^{24}K^6q^2Q^4\beta^{24} \ldots\ldots (45)$$

$$Z_3 = 3.645 * 10^{-22}a^{12}K^3Q^3\beta^{12} \ldots\ldots (46)$$

$$\zeta_1 = \frac{1}{a^4KQ\beta^4} \ldots\ldots (47)$$

Thus the expressions for A_{12}, A_{21}, and A_{22} are determined using the same procedure.

Determination of Stiffness and Mass Matrices

During natural vibrations, there exist modal interactions among different modes. Consequently in this work, the mode shape *w* and the weighting function *V* are interactively combined to effect these modal interactions. Thus to determine the natural frequencies, the expressions representing the energy functional are as given in Eqs. (19) and (20) are invoked and modified in accordance with these modal interactions as expressed in Eqs. (48) and (49) respectively.

$$\int_0^a\int_0^b\left[\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial x^2}+2\frac{\partial^2 V}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial y^2}+2KF_VF_w\frac{\partial^2 V}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+KF_VF_w\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+KF_VF_w\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]dxdy = \omega^2 MK\int_0^a\int_0^b Vwdxdy \quad (48)$$

$$\int_0^a\int_0^b\left[\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 w}{\partial x^2}+2\frac{\partial^2 V}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial y^2}+2KF_VF_w\frac{\partial^2 V}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+KF_VF_w\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right]dxdy = \omega^2 MK\int_0^a\int_0^b Vwdxdy \quad (49)$$

Where F_V and F_w are the stress functions associated with mode shapes corresponding to weighting function and displacement function respectively. For instance, to determine the stiffness component k_{11} at first mode, let $V = w = w(1,1)$; to determine the stiffness component $k_{21} = k_{12}$, let $V = w(1,2)$ and $w = w(1,1)$; to determine the stiffness component $k_{31} = k_{13}$, let $V = w(2,1)$ and $w = w(1,1)$; to determine the stiffness component $k_{41} = k_{14}$, let $V = w(2,2)$ and $w = w(1,1)$ and so on. For instance, the mode shapes for CCCC plate are as stated in Eqs. (50) through (53).

$$w(1,1) = A_{11}\left[\left(\frac{x}{a}\right)^2-2\left(\frac{x}{a}\right)^3+\left(\frac{x}{a}\right)^4\right]\left[\left(\frac{y}{b}\right)^2-2\left(\frac{y}{b}\right)^3+\left(\frac{y}{b}\right)^4\right] \quad (50)$$

$$w(1,2) = A_{12}\left[\left(\frac{x}{a}\right)^2-2\left(\frac{x}{a}\right)^3+\left(\frac{x}{a}\right)^4\right]\left[\left(\frac{y}{b}\right)^3-2\left(\frac{y}{b}\right)^4+\left(\frac{y}{b}\right)^5\right] \quad (51)$$

$$w(2,1) = A_{21}\left[\left(\frac{x}{a}\right)^3-2\left(\frac{x}{a}\right)^4+\left(\frac{x}{a}\right)^5\right]\left[\left(\frac{y}{b}\right)^2-2\left(\frac{y}{b}\right)^3+\left(\frac{y}{b}\right)^4\right] \quad (52)$$

$$w(2,2) = A_{22}\left[\left(\frac{x}{a}\right)^3-2\left(\frac{x}{a}\right)^4+\left(\frac{x}{a}\right)^5\right]\left[\left(\frac{y}{b}\right)^3-2\left(\frac{y}{b}\right)^4+\left(\frac{y}{b}\right)^5\right] \quad (53)$$

Then the stiffness matrix for CCCC rectangular plate is determined as follows:

$$\int_0^a\int_0^b\left[\left(\frac{\partial^2 w(1,1)}{\partial x^2}\right)^2+2\left(\frac{\partial^2 w(1,1)}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 w(1,1)}{\partial y^2}\right)^2+2K(F_{11})^2\left(\frac{\partial^2 w(1,1)}{\partial x\partial y}\right)^2+2K(F_{11})^2\left(\frac{\partial^2 w(1,1)}{\partial x^2}\right)\left(\frac{\partial^2 w(1,1)}{\partial y^2}\right)\right]dxdy = k_{11} \quad (54)$$

$$\int_0^a\int_0^b\left[\frac{\partial^2 w(1,2)}{\partial x^2}\frac{\partial^2 w(1,1)}{\partial x^2}+2\frac{\partial^2 w(1,2)}{\partial x\partial y}\frac{\partial^2 w(1,1)}{\partial x\partial y}+\frac{\partial^2 w(1,2)}{\partial y^2}\frac{\partial^2 w(1,1)}{\partial y^2}+2KF_{12}F_{11}\frac{\partial^2 w(1,2)}{\partial x\partial y}\frac{\partial^2 w(1,1)}{\partial x\partial y}+KF_{12}F_{11}\frac{\partial^2 w(1,2)}{\partial y^2}\frac{\partial^2 w(1,1)}{\partial x^2}+KF_{12}F_{11}\frac{\partial^2 w(1,2)}{\partial x^2}\frac{\partial^2 w(1,1)}{\partial y^2}\right]dxdy = k_{21} = k_{12} \quad (55)$$

$$\int_0^a\int_0^b\left[\frac{\partial^2 w(2,1)}{\partial x^2}\frac{\partial^2 w(1,1)}{\partial x^2}+2\frac{\partial^2 w(2,1)}{\partial x\partial y}\frac{\partial^2 w(1,1)}{\partial x\partial y}+\frac{\partial^2 w(2,1)}{\partial y^2}\frac{\partial^2 w(1,1)}{\partial y^2}+2KF_{21}F_{11}\frac{\partial^2 w(2,1)}{\partial x\partial y}\frac{\partial^2 w(1,1)}{\partial x\partial y}+KF_{21}F_{11}\frac{\partial^2 w(2,1)}{\partial y^2}\frac{\partial^2 w(1,1)}{\partial x^2}+KF_{21}F_{11}\frac{\partial^2 w(2,1)}{\partial x^2}\frac{\partial^2 w(1,1)}{\partial y^2}\right]dxdy = k_{31} = k_{13} \quad (56)$$

$$\int_0^a\int_0^b\left[\frac{\partial^2 w(2,2)}{\partial x^2}\frac{\partial^2 w(1,1)}{\partial x^2}+2\frac{\partial^2 w(2,2)}{\partial x\partial y}\frac{\partial^2 w(1,1)}{\partial x\partial y}+\frac{\partial^2 w(2,2)}{\partial y^2}\frac{\partial^2 w(1,1)}{\partial y^2}+2KF_{22}F_{11}\frac{\partial^2 w(2,2)}{\partial x\partial y}\frac{\partial^2 w(1,1)}{\partial x\partial y}+KF_{22}F_{11}\frac{\partial^2 w(2,2)}{\partial y^2}\frac{\partial^2 w(1,1)}{\partial x^2}+KF_{22}F_{11}\frac{\partial^2 w(2,2)}{\partial x^2}\frac{\partial^2 w(1,1)}{\partial y^2}\right]dxdy = k_{41} = k_{14} \quad (57)$$

$$\int_0^a\int_0^b\left[\left(\frac{\partial^2 w(2,2)}{\partial x^2}\right)^2+2\left(\frac{\partial^2 w(2,2)}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 w(2,2)}{\partial y^2}\right)^2+2K(F_{22})^2\left(\frac{\partial^2 w(2,2)}{\partial x\partial y}\right)^2+2K(F_{22})^2\left(\frac{\partial^2 w(2,2)}{\partial x^2}\right)\left(\frac{\partial^2 w(2,2)}{\partial y^2}\right)\right]dxdy = k_{44} \quad (58)$$

The resulting stiffness matrix is given as expressed in Eq. (59).

$$[\boldsymbol{K}] = \begin{bmatrix} k_{11}k_{12}k_{13}k_{14} \\ k_{21}k_{22}k_{23}k_{24} \\ k_{31}k_{32}k_{33}k_{34} \\ k_{41}k_{42}k_{43}k_{44} \end{bmatrix} \quad (59)$$

The same procedure is used to develop other required stiffness matrices for CCCS and CCCF respectively. In similar approach, the right hand side (RHS) of Eqs. (48) and (49) respectively will yield the mass matrices after due integration process. Thus, for CCCC rectangular plate, the mass matrix is determined as follows:

$$\omega^2 MK \int_0^a \int_0^b \left[\left(w(1,1)\right)^2\right] dxdy = b_{11} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (60)$$

$$\omega^2 MK \int_0^a \int_0^b [w(1,2) * w(1,1)] dxdy = b_{21} = b_{12} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (61)$$

$$\omega^2 MK \int_0^a \int_0^b [w(2,1) * w(1,1)] dxdy = b_{31} = b_{13} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (62)$$

$$\omega^2 MK \int_0^a \int_0^b \left[\left(w(2,2)\right)^2\right] dxdy = b_{44} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (63)$$

The resulting mass matrix is as given in Eq. (64).

$$[\boldsymbol{B}] = \omega^2 \begin{bmatrix} b_{11} b_{12} b_{13} b_{14} \\ b_{21} b_{22} b_{23} b_{24} \\ b_{31} b_{32} b_{33} b_{34} \\ b_{41} b_{42} b_{43} b_{44} \end{bmatrix} MK \ldots\ldots\ldots\ldots\ldots\ldots\ldots (64)$$

The same approach is used to develop the mass matrices for CCCS and CCCF respectively.

NUMERICAL EVALUATION AND DISCUSSION OF RESULTS

Natural vibrations of rectangular isotropic plates under geometrical nonlinearity are studied in this paper using weak-form variational principle. Numerical evaluations are required to facilitate the validation of the results of the present study with the results found in existing literature. Consequently the plate parameters used by [15] in his theoretical and experimental studies are adopted. The plate properties and dimensions are: E = 69GPa, ρ = 2700kgm^{-3}, μ = 0.33, a = 0.515m, t = 0.0003m, b = open, and $\beta = {}^{b}/_{a}$. For instance at aspect ratio, $\beta = 1$, these parameters are appropriately substituted and the numerical values for the elements in the stiffness matrix of Eq. (59) are as shown in Eq. (65).

$$[\boldsymbol{K}] = \begin{bmatrix} 0.0134702 & 0.00989551 & 0.00989551 & 0.00748755 \\ 0.00989551 & 0.0100967 & 0.00728298 & 0.00764936 \\ 0.00989551 & 0.00728298 & 0.0100967 & 0.00764936 \\ 0.00748755 & 0.00764936 & 0.00764936 & 0.00772896 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (65)$$

Similarly by substituting these parameters appropriately, the numerical values for the elements in the mass matrix of Eq. (64) are as shown in Eq. (66).

$$[\boldsymbol{B}] = \omega^2 \begin{bmatrix} 3.3713 & 2.48615 & 2.48615 & 1.88507 \\ 2.48615 & 2.00008 & 1.83341 & 1.51651 \\ 2.48615 & 1.83341 & 2.00008 & 1.51651 \\ 1.88507 & 1.51651 & 1.51651 & 1.25439 \end{bmatrix} 10^{-6} \ldots\ldots\ldots\ldots\ldots\ldots (66)$$

In the matrix manipulation, suppose the difference between $[\boldsymbol{K}]$ and $[\boldsymbol{B}]$ is $[\boldsymbol{A}]$, then

$$[\boldsymbol{A}] = [\boldsymbol{K}] - [\boldsymbol{B}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (67)$$

The determinant of $[\boldsymbol{A}]$ is thus:

$$det([\boldsymbol{A}]) = det([\boldsymbol{K}] - [\boldsymbol{B}]) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (68)$$

For nontrivial solution of the eign-function, Eq. (68) is set equal to zero. That is:

$$det([\boldsymbol{K}] - [\boldsymbol{B}]) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (69)$$

Eq. (69) can be numerically simplified by pre-multiplying Eq. (69) by the inverse of Eq. (66) as shown in Eq. (70).

$$det([\boldsymbol{B}]^{-1}[\boldsymbol{K}] - [\boldsymbol{B}]^{-1}[\boldsymbol{B}]) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (70)$$

The result of the manipulation of Eq. (70) gives an expression as shown in Eq. (71).

$$det\begin{bmatrix}(5184.69-R) & -9068.84 & -9068.84 & 3631.87\\ -1384.51 & (16365.3-R) & -516.634 & -14681.0\\ -1384.51 & -516.634 & (16365.3-R) & -14681.0\\ 1525.28 & 566.032 & 566.032 & (36201.1-R)\end{bmatrix}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (71)$$

Where

$$R=\omega^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (72)$$

The solution of the resulting quartic algebraic equation in terms of R and subsequently in terms of ω gives the natural frequencies of vibration of the rectangular CCCC plate at aspect ratio equal to one. Therefore by carrying out the same procedure for CCCS and CCCF rectangular plates respectively yields the desired natural frequencies for the rectangular plates up to first four lowest natural frequencies. Thus Table-1 presents the numerical values of nonlinear natural frequencies for CCCC, CCCS and CCCF rectangular plates based on the given plate parameters [15] at various aspect ratios.

Table-1: The numerical Values of Nonlinear Natural Frequencies of Rectangular Isotropic Plates at Various Aspect Ratios

Boundary Condition	Mode	Nonlinear Natural Frequency, (rad/sec)						
		Aspect Ratio, β						
		β = 0.25	β = 0.36	β = 0.50	β = 0.75	β = 1.0	β = 1.25	β = 1.50
CCCC	ω_{11}	640.425	320.960	172.397	89.322	63.206	53.747	50.670
	ω_{12}	676.967	363.289	222.893	150.737	129.930	93.965	76.401
	ω_{21}	1775.66	877.457	456.279	213.436	130.554	123.649	122.320
	ω_{22}	1816.94	920.843	503.190	267.272	190.506	159.849	147.133
CCCS	ω_{11}	449.157	229.946	128.671	73.042	56.515	51.350	50.012
	ω_{12}	497.906	284.272	189.583	140.704	117.846	87.672	73.263
	ω_{21}	1526.910	756.959	396.039	188.283	127.158	123.602	123.294
	ω_{22}	1574.730	806.889	449.375	247.901	183.338	157.669	147.266
CCCF	ω_{11}	151.804	88.512	60.912	47.074	43.086	41.490	40.706
	ω_{12}	213.099	154.296	129.791	102.302	73.586	60.936	54.549
	ω_{21}	652.499	335.338	187.077	117.660	114.008	112.441	111.607
	ω_{22}	731.443	414.379	265.742	179.608	149.568	136.017	129.167

From Table-1, it was observed that the nonlinear natural frequencies for all the sets of boundary conditions exhibit soft-spring type with respect to plate aspect ratios. Furthermore, it was observed that at very low aspect ratio, the fundamental nonlinear frequency approaches its equivalent beam fundamental frequency.

Also, in this study, the variations of the ratios of nonlinear natural frequencies to linear natural frequencies with plate aspect ratios were examined; and these variations are presented in Table-2.

Let $\lambda=\frac{\omega_{NL}}{\omega_L}$ = ratio of nonlinear natural frequency to linear natural frequency

Let β = Aspect ratio

Table-2: Variation of the Ratios of Nonlinear to Linear Natural Frequencies with Aspect Ratios

Boundary Condition	Ratio $\left(\frac{\omega_{NL}}{\omega_L}\right)$	β = 0.25	β = 0.50	β = 0.75	β = 1.00	β = 1.25	β = 1.50
CCCC	λ_{11}	1.00000	1.00000	1.00008	1.00406	1.02760	1.07136
	λ_{12}	1.00000	1.00000	1.00005	1.00010	1.01530	1.04279
	λ_{21}	1.00000	1.00000	1.00006	1.00490	1.01620	1.03946
	λ_{22}	1.00000	1.00000	1.00007	1.00328	1.01828	1.04663
CCCS	λ_{11}	1.00000	1.00000	1.00084	1.01519	1.05539	1.10451
	λ_{12}	1.00000	1.00000	1.00061	1.00772	1.02912	1.06415
	λ_{21}	1.00000	1.00000	1.00053	1.01062	1.03318	1.05840
	λ_{22}	1.00000	1.00000	1.00075	1.01103	1.03112	1.06362
CCCF	λ_{11}	1.00000	1.00000	1.00004	1.00007	1.00000	1.00000
	λ_{12}	1.00000	1.00000	1.00022	1.00186	1.00663	1.01517
	λ_{21}	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
	λ_{22}	1.00000	1.00000	1.00025	1.00184	1.00741	1.01261

From Table-2, it was observed that the ratios of nonlinear natural frequencies to linear natural frequencies with respect to aspect ratios exhibit hard-spring type. It was also observed from Table-2 that at the regime of aspect ratio less than or equal to 0.5, the linear natural frequency equals the corresponding nonlinear natural frequency. For the three sets of boundary conditions investigated, the observation is almost the same. Therefore it can be inferred from this observation that at aspect ratio less than 0.5, the dynamic analysis of rectangular isotropic plates of boundary conditions CCCC, CCCS and CCCF can be done with linear mathematical model without loss in accuracy.

The numerical value of the fundamental nonlinear frequency of CCCC rectangular isotropic plate evaluated at aspect ratio equal to 0.36 was used to validate the efficacy of the present study's approximation technique. The numerical value (320.96 rad/sec) obtained was compared with the previous result (321.071 rad/sec) due to [15] evaluated experimentally. The percentage error is 0.034. This comparison in view of statistical interpretation shows that the present study's approximation technique provides very satisfactory approximation to any engineering precision.

CONCLUSION

This paper utilizes weak-form variational principle and algebraic orthogonal polynomial displacements based on static deflections of orthogonal beam network to determine the nonlinear natural frequencies of highly clamped rectangular isotropic plates. The nonlinear natural frequencies computed in this work, in view of statistical interpretation, are in excellent agreement with results from previous works found in literature. Therefore, it is here-upon concluded that the application of weak-form variational principle in dynamic analysis has high competitive capability with other approximation techniques usually employ in problems of dynamic analysis of rectangular isotropic plates. Also, it is here-unto concluded that the application of algebraic polynomial displacement functions is capable and stable in yielding satisfactory approximations to any set of boundary conditions of rectangular plates. Furthermore, it was observed that at plate aspect ratio less than or equal to 0.5, the linear natural frequencies approximate the nonlinear natural frequencies; and therefore it is here concluded that the linearized mathematical model can be employed in place of nonlinear mathematical model within the regime of low aspect ratios without loss in accuracy.

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