OPEN ACCESS
Saudi Journal of Engineering and Technology
Abbreviated Key Title: Saudi J Eng Technol
ISSN 2415-6272 (Print) |ISSN 2415-6264 (Online)
Scholars Middle East Publishers, Dubai, United Arab Emirates
Journal homepage: http://scholarsmepub.com/sjet/

Review Article

Analysis and Comparative Study of First and Second Order Runge Kutta Method Using MATLAB

Nikzad Jamali*
Assistant Professor at Sarepul Higher Education Institute, Afghanistan

***Corresponding author:** Nikzad Jamali
| **Received:** 03.04.2019 | **Accepted:** 09.05.2019 | **Published:** 14.05.2019
DOI:10.21276/sjeat.2019.4.5.3

Abstract

This paper presents a brief work about numerical method to solve ODE with IVP. In this paper firstly derived the first and second order R-K method, than a comparative study will be done use the MATLAB program by taking an example, also in this paper we will show that the second order R-K method is more accurate than first order. Also in this paper present that the accuracy of the methods depend to step size, if the step size taken small then all the numerical methods will be convergence.
Keywords: Numerical methods, ODE, IVP, MATLAB.

INTRODUCTION

Since the basis and laws of nature are dependent on calculation, mathematics in this field has a certain cost and every law of nature results in the formation of a differential equation, and solution of differential equations is one of the foundations of mathematical skill, which various methods have been discovered by scientists, but we cannot calculate all differential equations in analytical methods. Here, numerical methods are introduced and provide solutions methods for complicated problem, so numerical method is mathematical model, which approximate the solution of a problem. In this section, it is necessary to introduce a number of elementary topics and definitions, then we will consider numerical methods later.

Since the main subject is differential equations, we start with the definition of differential equations. The relation between the independent variable, the function and the derivatives of the function is called differential equations and the highest order of derivatives in the equation is called the order of equation, and the degree of highest order of derivatives is called degree of equations.

Historically, in 17^{th} century "Isaac Newton" (1642-1727) and "Gottfried W. Leibniz" (1646-1916) discovered diff equation, in 18^{th} century Newton's development of the calculus and principle of mechanic's provided a main basis for the uses. He classified the first order differential equations in the formula of $\frac{dy}{dx}=\varphi(x)$, $\frac{dy}{dx}=\varphi(y)$, $\frac{dy}{dx}=\varphi(x,y)$. Later he also classified a method of solution called infinite series [1, 2]. The brothers Jakobe and Johann also did a great contribution in the discovery of differential equation. Jakobe, "Bernoulli" resolved differential equation $y'=[\alpha^3 / (\beta^2 y-\alpha^3)]^{\frac{1}{2}}$ and Johann solved $dy/dx = y/ax$. Leibnitz discovered the method of separation of variable reduction of homogenous equation into the separable form and also discover the way of solving first order linear equation. He also solved the problem called the Brachistochrone problem. One of the works of "Daniel Bernoulli" (1700- 1782), (son of Johann) is to introduce the Bessel function. One of the students of Johann Bernoulli who was the great mathematician all the time in 18^{th} century namely Leonhard Euler (1707-1785). His interest are collectively distributed in almost all areas of mathematics. One of his interest in the foundation of problem in mechanics in the language of mathematics. Euler had also worked on the exactness of first order differential equation. In (1734-1735) he developed, the integrating factors theory and provide the solution of homogenous linear equation in 1743. Later also worked on nonhomogeneous equation in 1750 - 51. In 1750 he uses power series for solving differential equation. In 1768–69 he also proposed numerical procedure. Laplace's equality is used in various branches of scientific physics. One of the term is Laplace transform which is by his name, is useful in solving differential equation. Many ways of solving ordinary differential equation has discovered in the end of 18^{th} century. In the past 50 years there is an increasing role in the development of versatile computers, has greater scope in the problems that can be studied by numerical methods. In the meantime, improved and robust numerical integrators have developed. In the 20^{th} century one of characteristic of differential equation is development of geometrical and topological methods the aim of this characteristic is to understand the qualitative behavior of solution in both the ways analytically as well as geometrically. If some more information needs, it is obtained using numerical estimates. In the last few years, the two things work together. Computer and the graphics of computers have given a new shape to the study of nonlinear differential equation.

Basic Theorems

Existence Theorem: Let $\varphi(x, y)$ be a continuous function at all point (x, y) in some rectangle:

$R: |x - x_0| < a \quad , \quad |y - y_0| < b$

And bounded in R. I.e $\exists\ K \in R$ such that $|\varphi(x, y)| \le K \quad \forall\ (x, y) \in R^2$ then the IVP $y' = \varphi(x, y)$, $y(x_0) = y_0$ has at least one solution $y(x)$, this solution exist at least for all x in the subinterval, $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$ where α is smaller of two numbers a and b/K .

Uniqueness Theorem: Let φ and its partial derivatives $\varphi_y = \dfrac{\partial \varphi}{\partial y}$ be continuous for all points in the rectangle R and bounded i.e $|\varphi(x, y)| \le K$ & $|\varphi_y(x, y)| \le M \quad \forall\ (x, y) \in R^2$ then the IVP $y' = \varphi(x, y)$ $y(x_0) = y_0$ has at most one solution $y(x)$.

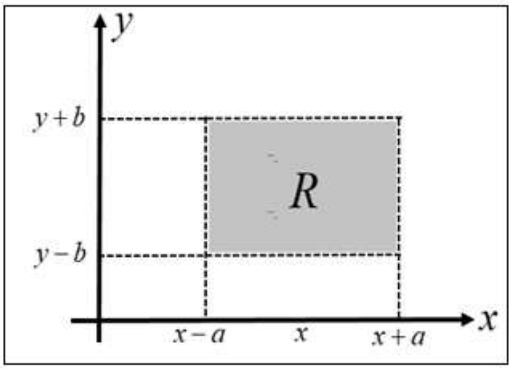

Fig-1

Single step methods

General form of this technique is $y_{j+1} = y_j + h\mu(x_j, y_j; h)$, in this method we use the point (x_j, y_j) to find (x_{j+1}, y_{j+1}), i.e for finding a desired value, we use only one previous amount of that value.

Some researchers have written very useful articles that the reader can use to make a great use of them, we will briefly introduce the following.

Ogunrinde R. Bosede [2] the author of this article presented the study of errors by presenting examples of solving ODE using IVP. And it has also been shown that the RK method is better than other methods. Steven C Chapra [3] describe the views about MATLAB program. This book discuses about MATLAB program to solve single step method with its initial and boundary value problem. This book is an applied book and author proposed use as a good reference in mathematic filed. Fadugba S. E *et al*., [4] this article is also about solving ODE using numerical methods. Here, the author first describes all methods, such as Euler's method, RK method, and makes it clear with numerous examples.

Md. Amirul Islam [5] This article is about IVP and here the author has methodically analyzed the subject with a number of examples. Parvesh Ranga [6]. In this paper, the subject of the Eller method is analyzed and the issue of errors is also examined C. Senthilnathan [7]. This is one of the great articles in the solution of ODE section. Here the subject of IVP is easily stated. The RK method is compared with several examples. The author has used the MATLAB program to clarify the subject, in this paper.

Runge Kutta Technique

Consider the general form of ODE with first order and first degree with an IVP $y(x_0) = y_0$.

$$\frac{dy}{dx} = \xi(x, y) \tag{2.1}$$

Now we use a numerical method to solve this difference equation. One of the method is one step method which we define as:

$$new\ value = old\ value + slope \times step\ size$$

Or In mathematics form we defined as:

$$y_{j+1} = y_j + \varphi h \tag{2.2}$$

According to equation (2.2), it shows that the slope estimate of φ is used to extrapolate from on old price y_j to a new price y_{j+1} according to a distance h, so the equation (2) used step by step to fine the desired solution.

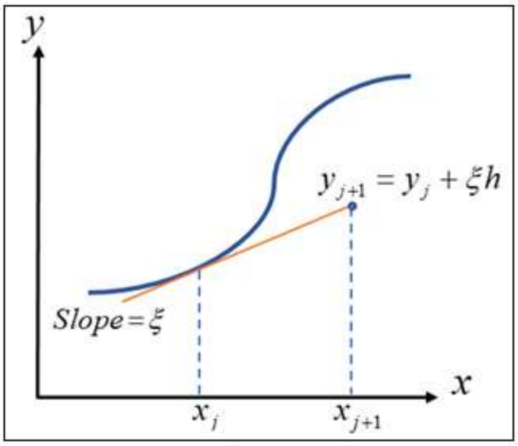

Fig-2

The whole single step method can be expressed in this general form, with only difference of estimating the slope. All these methods are generally called RK technique and defined as

$$y_{j+1} = y_j + h(w_1u_1 + w_2u_2 + \ldots w_m u_m)$$
$$whrer \quad w_1 + w_2 + \ldots w_m = 1$$
$$u_1 = \xi(x_j, y_j)$$
$$u_2 = \xi(x_j + \alpha_1 h, y_j + \beta_1 u_1 h)$$
$$u_3 = \xi(x_j + \alpha_2 h, y_j + \beta_{21} u_1 h + \beta_{22} u_2 h)$$
$$\vdots$$
$$u_m = \xi(x_j + \alpha_{m-1} h, y_j + \beta_{m-1} u_1 h + \beta_{m-2} u_2 h + \ldots + \beta_{m-1} u_{m-1} h)$$

First Order RK Method

The general form of this method is

$$y_{j+1} = y_j + h w_1 u_1$$

By definition of general form we have $w_1 = 1$ and $u_1 = \xi(x_j, y_j)$, then the first order R-K method is defined as

$$y_{j+1} = y_j + h\xi(x_j, y_j) \qquad 2.3$$

The equation (2.3) also known as Euler method
This formula also derived by Taylor series:

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2}y''(x) + ... \qquad 2.4$$

If h is small, then we can neglect the higher power of h, then the equation (2.4) approximately can be written as

$$y(x+h) = y(x) + hy'(x)$$

But by definition we have $y'(x) = \xi(x_j, y_j)$, now we can write

$$y(x+h) = y(x) + h\xi(x_j, y_j)$$

or

$$y_{j+1} = y_j + h\xi(x_j, y_j)$$

Note: in this method as we truncate the $\frac{h^2}{2!}$ and also the higher terms, so this method is of order two.

It means that the truncation error in this method is of order tow which we denoted $T = O(h^2)$.

Second Order R-K Techniques

General form of this method is

$$y_{j+1} = y_j + h(w_1u_1 + w_2u_2) \qquad 2.5$$

Where $u_1 = \xi(x_j, y_j)$ *and* $u_2 = \xi(x_j + \alpha_1 h, y_j + \beta_1 u_1 h)$

Note: for further calculation we need the Taylor series for a two variable function which is denoted as

$$\varphi(x+r, y+s) = \varphi(x,y) + r\frac{\partial\varphi}{\partial x} + s\frac{\partial\varphi}{\partial y} + ...$$

After neglecting the higher order we can write

$$\varphi(x+r, y+s) = \varphi(x,y) + r\frac{\partial\varphi}{\partial x} + s\frac{\partial\varphi}{\partial y} + O(h^2)$$

Now expand the value of u_2 by Taylor series

$$k_2 = \xi(x_j + \alpha_1 h + y_j + \beta_1 u_1 h) = \xi(x_j, y_j) + \alpha_1 h\frac{\partial f}{\partial x} + \beta_1 u_1 h\frac{\partial f}{\partial y} + O(h^2)$$

Now put the value of u_1 and u_2 in equation (2.5), then we get

$$y_{j+1} = y_j + h[w_1\xi(x_j, y_j) + w_2\xi(x_j, y_j)] + w_2\alpha_1 h\frac{\partial\xi}{\partial x} + w_2\beta_1 h\xi(x_j, y_j)\frac{\partial\xi}{\partial y} + O(h^2)$$

Or simply we can write

$$y_{j+1} = y_j + (w_1 + w_2)h\xi + h^2(w_2\alpha_1\xi_x + w_2\beta_2\xi\cdot\xi_y) + O(h^2)) \qquad 2.6$$

Now for finding the value of $w_1, w_2, \alpha_1, \beta_2$ we should find the exact solution by Taylor series. Consider the first order and first degree of ODE with first and second derivatives

$$y' = \xi(x,y) \qquad \& \qquad y'' = \frac{\partial\xi}{\partial x} + \xi\cdot\frac{\partial\xi}{\partial y}$$

Now Taylor series determine as

$$y_{j+1} = y_j + hy' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y''' + \ldots$$

Or we can write

$$y_{j+1} = y_j + h\xi + \frac{h^2}{2!}(\xi_x + \xi \cdot \xi_y) + O(h^3) \qquad 2.7$$

By comparing the coefficient of h, h^2 in equation (2.6) and (2.7), we obtained the below system of equation

$$w_1 + w_2 = 1$$

$$w_2\alpha_1 = \frac{1}{2}$$

$$w_2\beta_2 = \frac{1}{2}$$

This system of equation is with (3) equation and four unknown so, one of this parameters are arbitrary so, we have

$$w_1 = 1 - w_2 \quad , \quad \alpha_1 = \beta_1 = \frac{1}{2w_2} \quad and \quad w_2 \neq 0$$

Now we can conclude that the second order of RK method is a family, which depended the value of w_2, by deferent value of w_2, we can defined deferent methods. The famous value of w_2 is $w_2 = 1$, $w_2 = \frac{1}{2}$, $w_2 = \frac{2}{3}$ which is known as: midpoint method, Heim's method and Ralston's, method respectively.

Midpoint Technique

Now if we take $w_2 = 1$ then we have $w_1 = 0$, $\alpha_1 = \beta_1 = \frac{1}{2}$ then we will obtain:

$$y_{j+1} = y_j + h[w_1\xi(x_j, y_j)] + w_2[\xi(x_j + \alpha_1 h, y_j + \beta_1 u_1 h)]$$

$$y_{j+1} = y_j + h[0 \cdot \xi(x_j, y_j) + 1 \cdot \xi(x_j + \frac{h}{2}, y_j + \frac{h}{2}u_1)]$$

$$y_{j+1} = y_j + h\xi(x_j + \frac{h}{2} \ , \ y_j + \frac{h}{2}\xi(x_j, y_j))$$

Heun's Technique

If $w_2 = \frac{1}{2}$ then $\alpha_1 = \beta_2 = 1$ and $w_1 = \frac{1}{2}$ now this method obtained as:

$$y_{j+1} = y_j + h(w_1 u_1 + w_2 u_2)$$

$$y_{j+1} = y_j + \frac{h}{2}(u_1 + u_2)$$

Where $u_1 = \xi(x_j, y_j)$ and $u_2 = \xi(x_j + h \ , \ y_j + u_1 h)$

Ralston's Technique

For $w_2 = \frac{2}{3}$ the value of w_1 , α_1 and β_1 will be achieve as $w_1 = \frac{1}{3}$, $\alpha_1 = \beta_1 = \frac{3}{4}$ then we have:

$$y_{j+1} = y_j + \frac{h}{3}(u_1 + 2u_2)$$

When $u_1 = \xi(x_j, y_j)$ and $u_2 = \xi(x_j + \frac{3}{4}h \ , \ y_j + \frac{3}{4}u_1 h)$

Example and Comparative Study

Example: Consider the first order and first degree ODE $y' = 4e^{0.8x} - 0.5y$ with initial condition $y(0) = 2$, the exact answer of this equation according to the given condition is $y = 2/1.3(e^{0.8x} - e^{-0.5}) + 2e^{-0.5x}$, find the value of y at $x = 1$.

Solution: we will solve this equation with two different step size, $h = 0.2$ & $h = 0.1$ after that we will compare the graphs, tables and get the results.

Table-1

x	Exact Answer	Euler's Method	Midpoint Method	Heun's Method	Ralston Method
0.0	2.00000	2.00000	2.000000	2.000000	2.000000
0.2	2.63636	2.60000	2.660630	2.663404	2.661998
0.4	3.35561	3.27881	3.404839	3.410595	3.407678
0.6	4.17473	4.05263	4.250202	4.259210	4.254645
0.8	5.11344	4.94023	5.217042	5.229641	5.223257
1.0	6.19463	5.96339	6.328929	6.345544	6.337125

Error table for h=0.2

Table-2

x	Exact Answer	Error of Euler's Method	Error of Midpoint Method	Error of Heun's Method	Error of Ralston Method
0.0	2.00000	0.00000	0.000000	0.000000	0.000000
0.2	2.63636	0.03636	0.024270	0.027044	0.025638
0.4	3.35561	0.07680	0.049229	0.054985	0.052068
0.6	4.17473	0.12210	0.075472	0.084480	0.079915
0.8	5.11344	0.17321	0.103602	0.116201	0.109817
1.0	6.19463	0.23124	0.134299	0.150914	0.142495

Figures for h=0.2

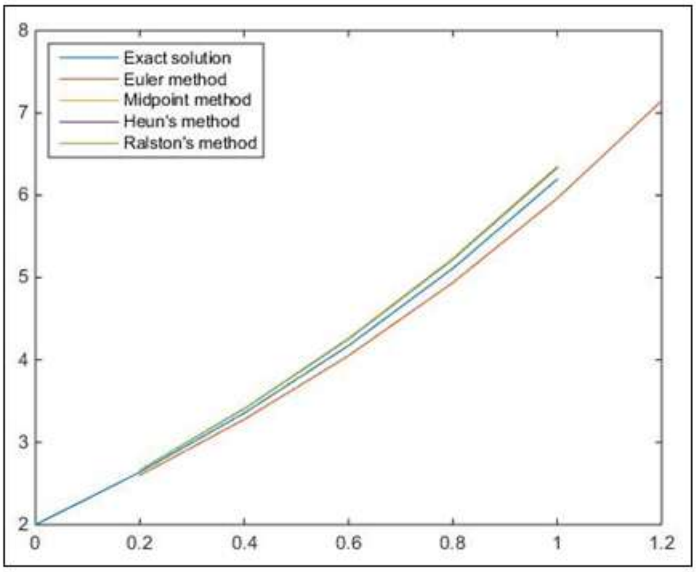

Fig-3

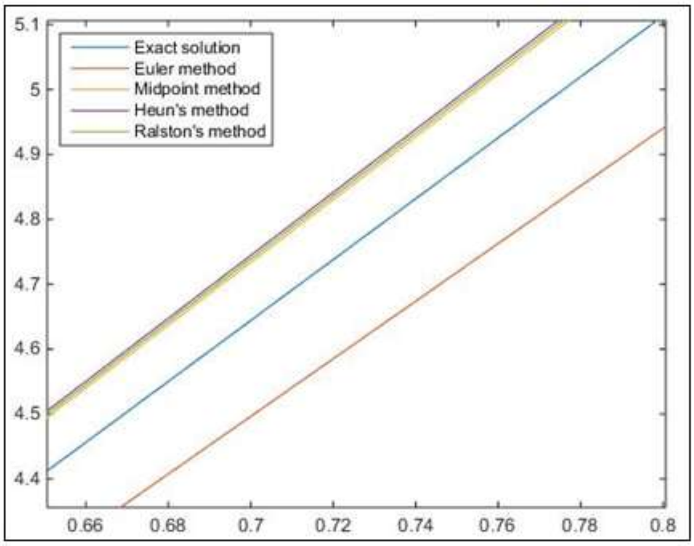

Fig-4

Average Error table of the Methods

Table-3

Average Error of Euler's Method	Average Error of Midpoint Method	Average Error of Heun's Method	Average Error of Ralston Method
0.127942	0.077374	0.086725	0.0819866

Solution table for h=0.1

Table-4

x	Exact Answer	Euler's Method	Midpoint Method	Heun's Method	Ralston Method
0.0	2.00000	2.00000	2.000000	2.000000	2.000000
0.1	2.30879	2.30000	2.315574	2.315907	2.315740
0.2	2.63636	2.61831	2.650000	2.650678	2.650337
0.3	2.98462	2.95680	3.005219	3.006254	3.005733
0.4	3.35561	3.31746	3.383315	3.384721	3.384013
0.5	3.75152	3.70244	3.786526	3.788321	3.787418
0.6	4.17473	4.11405	4.217265	4.219467	4.218358
0.7	4.62779	4.55478	4.678123	4.680754	4.679430
0.8	5.11344	5.02731	5.171899	5.174982	5.173430
0.9	5.63466	5.53453	5.701605	5.705166	5.703373
1.0	6.19463	6.07958	6.270494	6.274561	6.272514

Error table for h=0.1

Table-5

x	Exact Answer	Error of Euler's Method	Error of Midpoint Method	Error of Heun's Method	Error of Ralston Method
0.0	2.00000	0.00000	0.000000	0.000000	0.000000
0.1	2.30879	0.00879	0.006784	0.007117	0.006950
0.2	2.63636	0.01805	0.013640	0.014318	0.013977
0.3	2.98462	0.02782	0.020599	0.021634	0.021113
0.4	3.35561	0.03815	0.027705	0.029111	0.028403
0.5	3.75152	0.04908	0.035006	0.036801	0.035898
0.6	4.17473	0.06068	0.042535	0.044737	0.043628
0.7	4.62779	0.07301	0.050333	0.052964	0.051640
0.8	5.11344	0.08613	0.058459	0.061542	0.059990
0.9	5.63466	0.10013	0.066945	0.070506	0.068713
1.0	6.19463	0.11505	0.075864	0.079931	0.077884

Average Error table of the Methods

Table-6

Average Error of Euler's Method	Average Error of Midpoint Method	Average Error of Heun's Method	Average Error of Ralston Method
0.057689	0.039787	0.0418661	0.0408196

Figures for h=0.1

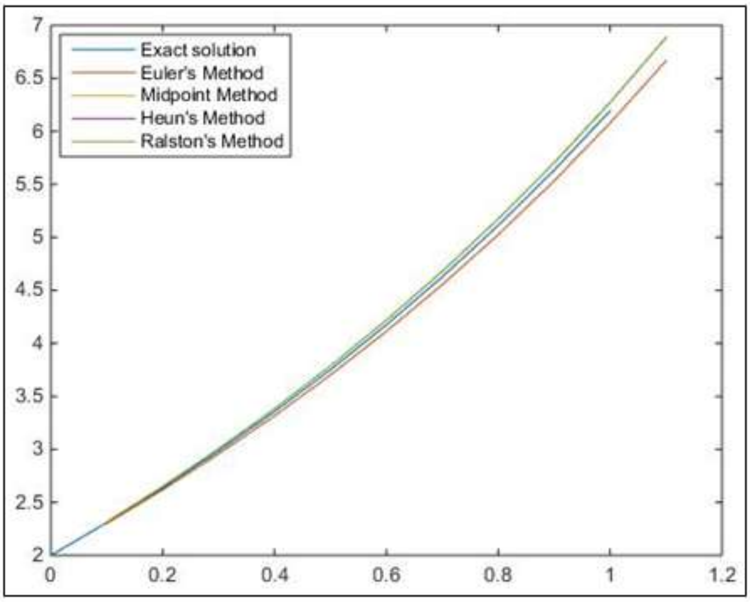

Fig-5

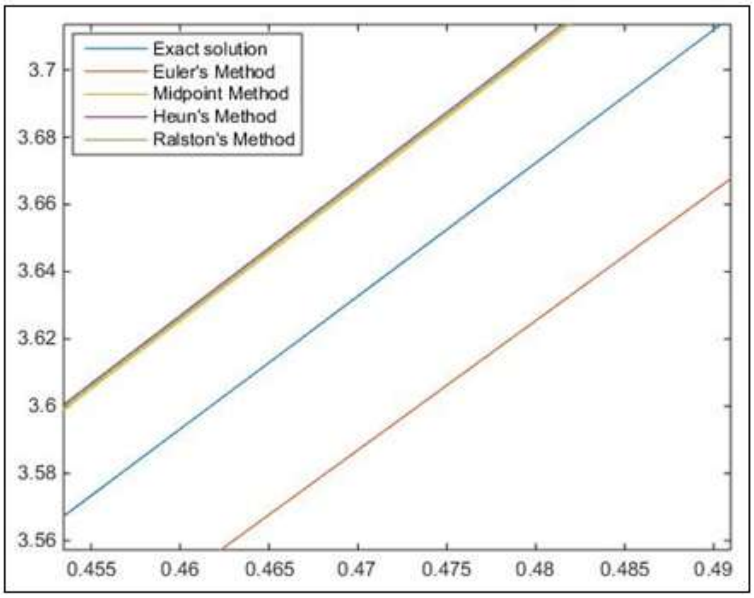

Fig-6

OUTCOMES

By considering the Table-2 and 5, which is called error table clearly we realized that the solution of single step methods directly dependent to value of h. whenever h=0.2 the methods are not more accurate, but for h=0.1 methods show more accuracy. We can realized same result in figure 1 and 3 graphically. Also by Table 3 and 6, we can realize, the second order RK method is better than first order and the Midpoint method is more than accurate from all mentioned methods.

CONCLUSION
In this paper the Range Kutta Technique for solving IVP has been studied. Two various methods namely, first and second order RK techniques described. By using MATLAB a comparative study has been done. From different tables and figures we conclude that, the step size directly effect in these methods. If we choose small step size, then every methods will have more accuracy. Another parameter which effect to convergence is, order of methods. Since the second order RK technique is, of order three and first order RK technique is, of order two, so the second order RK technique is more accurate than first order.

ACKNOWLEDGEMENT
I would like to express my special thanks and appreciation to Dr. Ashok Pal Professor and HOD of Mathematics of University Institute of Science, Chandigarh University, who have guided me to complete this paper. I would like to thank Dr. Kuldip Katiyar Assistant professor of Mathematics Department, University Institute of Science, Chandigarh University, for his support, in writing MATLAB program.

REFERENCES
1. Fadugba, S. E., & Nwozo, C. R. (2010). On the Comparative Study of Some Numerical Methods for the Solution of initial value problems in ODE. *International Journal of Innovation in Science and Mathematics*, 2, 61-67.
2. Ogunrinde, R. B., Fadugba, S. E., & Okunlola, J. T. (2012). On some numerical methods for solving initial value problems in Ordinary Differential Equations. *On Some Numerical Methods for Solving Initial Value Problems in Ordinary Differential Equations*.
3. Chapra, S. C. (2012). Applied Numerical Methods with MATLAB for engineering and science, McGraw Hill, third Edition.
4. Fadugba, S. E., & Nwozo, C. R. (2014). On the Comparative Study of Some Numerical Methods for Vanilla Option Valuation. *Communications in Applied Sciences*, *2*(1), 61-67.
5. Islam, M. A. (2015). Accuracy Analysis of Numerical solutions of Initial Value Problems (IVP) for Ordinary Differential Equations (ODE). *IOSR Journal of Mathematics*, *11*, 18-23.
6. Parvesh, R. (2017). An Analysis of Numerical Solutions and Errors with Euler's Method. *Global Journal for Research Analysis*, 6, 551-552.
7. Senthilnathan, C. (2018). A Numerical Solution of Initial value problem For ODE with Euler and Higher Order of Runge Kutta Methods using MATLAB. *International Journal of Engineering Science Invention*, 7, 25-31.