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Weighted Exponential - G Family of Probability Distributions

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Abstract: Many generalized families of distributions have been proposed and studied over the last two decades for modeling data in many applied areas. In this paper, depending on the idea of Bourguignon et al., a new family of weighted exponential distribution named as weighted exponential G −family of probability distributions has been proposed and studied which based on using weighted exponential generator. In addition, weighted exponential - uniform and weighted exponential - Kumaraswamy distributions have been provide as two examples for this new family.

Keywords: Weighted Exponential Distribution; G − family of probability distributions; Maximum Likelihood Estimation.

INTRODUCTION

The weighted exponential, (WE) distribution has been introduced by Gupta and Kundu [1]. Many generalized families of this distribution have been proposed and studied. Al-Mutairi *et al.*, [2] investigated the extension of the weighted exponential distribution to the bivariate and multivariate cases. Roy and Adnan [3] introduced a new class of circular distribution which is called wrapped weighted exponential distribution. Zamani *et al.*, [4] presented a new class named bivariate Poisson weighted exponential distribution and studied several properties, such as, mean, variance, correlation and joint moment generating function. Oguntunde [5] suggested a generalization of the weighted exponential distribution using the exponentiated class of distribution.

Oguntunde *et al.*, [6] introduced and provided the basic mathematical properties of a new class of two parameters weighted exponential distribution. Mahdavi and Jabbari [7] proposed a new class of weighted distributions by incorporating an extended exponential distribution in Azzalini's method. In this paper, we are interested in proposed a new family of weighted exponential distribution named as weighted exponential G −family of probability distributions based on using weighted exponential generator.

The WE distribution has the probability density function whose shape is very similar to the shape of probability density function of Weibull, gamma and generalized exponential distributions, while the WE distribution has many good properties and can be applied as a good fit for modeling life time data [8]. The probability density function (pdf) of WE distribution is given by [9]:

$$f_{WE}(x;\alpha,\lambda)=\frac{\alpha+1}{\alpha}\,\lambda\,e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\;;\;x>0\;;\;\alpha,\lambda>0 \qquad \dots(1)$$

The corresponding cumulative distribution function of WE distribution is given by [10]:

$$F_{WE}(x;\alpha,\lambda)=1-\frac{1}{\alpha}\,e^{-\lambda x}\left(\alpha+1-e^{-\alpha\lambda x}\right) \qquad \dots(2)$$

The reliability and hazard functions of WE distribution at time (t), respectively, are given by [11]:

$$R_{WE}(t;\alpha,\lambda)=1-F(t;\alpha,\lambda)=\frac{1}{\alpha}\,e^{-\lambda t}\left(\alpha+1-\,e^{-\alpha\lambda t}\right) \qquad \dots(3)$$

$$h_{WE}(t;\alpha,\lambda)=\frac{f(t;\alpha,\lambda)}{R(t;\alpha,\lambda)}=\frac{(\alpha+1)\,\lambda\left(1-e^{-\alpha\lambda t}\right)}{\alpha+\,1\,-\,e^{-\alpha\lambda t}} \qquad \dots(4)$$

The r^{th} moments about the origin is [12]:

$$E(X^r) = \frac{(\alpha+1)\,\Gamma(r+1)}{\alpha\,\lambda^r}\left(1-\frac{1}{(1+\alpha)^{r+1}}\right)\;;\; r = 1,2,3,\ldots \qquad \ldots(5)$$

The moment generating function, $M_X(t)$, for $-1 < t < 1$ can be expressed by [13]:

$$M_X(t) = E(e^{tX}) = \frac{(\alpha+1)\lambda}{\alpha}\left[\frac{1}{\lambda-t} - \frac{1}{\lambda-t+\lambda\alpha}\right] \qquad \ldots(6)$$

In particular:

$$M'_X(0) = E(X) = \frac{\alpha+2}{\lambda(\alpha+1)} \qquad \ldots(7)$$

$$M''_X(0) = E(X^2) = \frac{2(\alpha^2+3\alpha+3)}{\lambda^2(\alpha+1)^2} \qquad \ldots(8)$$

Then,

$$v(X) = \frac{1}{\lambda^2}\left[1+\frac{1}{(\alpha+1)^2}\right] \qquad \ldots(9)$$

Weighted Exponential - G Family of Probability Distributions

Bourguignon *et al.*, [14] proposed and studied a new family of Weibull distribution based on using Weibull generator and they called it $Weibull - G$ family of probability distributions. The term generator means that for each parent (baseline) distribution we have a different cumulative distribution function. However, depending on their idea and to follow the same way, we will present a new family to WE distribution. Additionally, we provide two examples of the mathematical proposed family of distributions.

Consider a continuous distribution G with density g of the WE distribution as in equation (1). Depending on this density by replacing x with $\frac{G(x;\xi)}{\bar{G}(x;\xi)}[\bar{G}(x;\xi) = 1 - G(x;\xi)]$, Such that $G(x;\xi)$ is a parent (baseline) cumulative distribution function depends on a parameter vector ξ, we define the cumulative distribution function family by:

$$F_{WE-G}(x;\alpha,\lambda,\xi) = \int_0^{\frac{G(x;\xi)}{\bar{G}(x;\xi)}} \frac{\alpha+1}{\alpha}\,\lambda\, e^{-\lambda u}\left(1-e^{-\alpha\lambda u}\right) du$$

$$= \frac{1}{\alpha}\left[(\alpha+1)\left(1-e^{-\frac{\lambda G(x;\xi)}{\bar{G}(x;\xi)}}\right) + e^{-\frac{\alpha\lambda(+1)G(x;\xi)}{\bar{G}(x;\xi)}} - 1\right]$$

$$F_{WE-G}(x;\alpha,\lambda,\xi) = 1 - \frac{1}{\alpha} e^{-\frac{\lambda G(x;\xi)}{\bar{G}(x;\xi)}}\left(\alpha+1-e^{-\frac{\alpha\lambda G(x;\xi)}{\bar{G}(x;\xi)}}\right) \qquad \ldots(10)$$

The family pdf is given by:

$$f_{WE-G}(x;\alpha,\lambda,\xi) = \frac{\partial F_{WE-G}(x;\alpha,\lambda,\xi)}{\partial x}$$

$$f_{WE-G}(x;\alpha,\lambda,\xi) = \frac{\lambda(\alpha+1)g(x;\xi)}{\alpha[\bar{G}(x;\xi)]^2}\; e^{-\frac{\lambda G(x;\xi)}{\bar{G}(x;\xi)}}\left(1-e^{-\frac{\alpha\lambda G(x;\xi)}{\bar{G}(x;\xi)}}\right) \qquad \ldots(11)$$

Then, a random variable X with pdf as in equation $(11)$is denote by $X \sim WE - G(\alpha,\lambda,\xi)$

The reliability and hazard functions of $WE - G$ family at time (t), respectively, are given by:

$$R_{WE-G}(t;\alpha,\lambda,\xi) = \frac{1}{\alpha} e^{-\frac{\lambda G(t;\xi)}{\bar{G}(t;\xi)}}\left(\alpha+1-e^{-\frac{\alpha\lambda G(t;\xi)}{\bar{G}(t;\xi)}}\right)$$

$$h_{WE-G}(t;\alpha,\lambda,\xi) = \frac{\lambda(\alpha+1)g(t;\xi)\left(1-e^{-\frac{\alpha\lambda G(t;\xi)}{\bar{G}(t;\xi)}}\right)}{[\bar{G}(t;\xi)]^2\left(\alpha+1-e^{-\frac{\alpha\lambda G(t;\xi)}{\bar{G}(t;\xi)}}\right)}$$

In the other side, in order to obtain some mathematical properties of the $WE - G$ distribution, we have to mention that the pdf of $WE - G$ in equation (11) can be expressed as an infinite linear combination of G distribution where the mathematical properties can be obtained directly from those properties. By using the power series for the exponential function, we get:

$$e^{-\frac{\lambda G(x;\xi)}{\bar{G}(x;\xi)}} = \sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k}{k!}\left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)^k \qquad \ldots(12)$$

$$e^{-\frac{\alpha\lambda G(x;\xi)}{\bar{G}(x;\xi)}} = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k \lambda^k}{k!} \left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)^k \quad \dots (13)$$

Inserting (12), (13) in (11), we get:

$$f_{WE-G}(x;\alpha,\lambda,\xi) = \frac{(\alpha+1)g(x;\xi)}{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{k+1} G(x;\xi)^k}{k!\left(\bar{G}(x;\xi)\right)^{k+2}} \left(1 - \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k \lambda^k}{k!} \left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)^k\right) \quad \dots (14)$$

Then, by using the generalization of Binomial theorem, we get:

$$\left(\bar{G}(x;\xi)\right)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{j!\ \Gamma(k)} \left(G(x;\xi)\right)^j \quad \dots (15)$$

$$\left(\bar{G}(x;\xi)\right)^{-(k+2)} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j+2)}{j!\ \Gamma(k+2)} \left(G(x;\xi)\right)^j \quad \dots (16)$$

Inserting (15), (16) in (14), we get:

$$f_{WE-G}(x;\alpha,\lambda,\xi) = \frac{(\alpha+1)g(x;\xi)}{\alpha} \sum_{k,j=0}^{\infty} \frac{(-1)^k \lambda^{k+1} \Gamma(k+j+2)}{k!\ j!\ \Gamma(k+2)} \left(G(x;\xi)\right)^{k+j} \left(1 - \sum_{k,j=0}^{\infty} \frac{(-1)^k \alpha^k \lambda^k \Gamma(k+j)}{k!\ j!\ \Gamma(k)} \left(G(x;\xi)\right)^{k+j}\right) \quad \dots (17)$$

Let:

$$h_{k,j} = \left(G(x;\xi)\right)^{k+j} \quad \dots (18)$$

$$w_{k,j} = \frac{(-1)^k \lambda^{k+1} \Gamma(k+j+2)}{k!\ j!\ \Gamma(k+2)} \quad \dots (19)$$

$$m_{k,j} = \frac{(-1)^k \alpha^k \lambda^k \Gamma(k+j)}{k!\ j!\ \Gamma(k)} \quad \dots (20)$$

Inserting (18), (19) and (20) in (17), we get:

$$f_{WE-G}(x;\alpha,\lambda,\xi) = \frac{(\alpha+1)\ g(x;\xi)}{\alpha} \sum_{k,j=0}^{\infty} w_{k,j}\ h_{k,j} \left(1 - \sum_{k,j=0}^{\infty} m_{k,j}\ h_{k,j}\right) \quad \dots (21)$$

In the following subsections, we give two examples of the $WE - G$ family of distributions call the weighted exponential - uniform distribution and the weighted exponential - Kumaraswamy distribution.

Weighted Exponential - Uniform Distribution

Consider the parent distribution is Uniform distribution with density and cumulative distribution functions, respectively, given by (Bourguignon et al., 2014):

$$g_U(x;\theta) = \frac{1}{\theta} \qquad 0 < x < \theta < \infty \quad \dots (22)$$

$$G_U(x;\theta) = \frac{x}{\theta} \qquad 0 < x < \theta < \infty \quad \dots (23)$$

Now, depending on equation (10), $\frac{G(x;\xi)}{\bar{G}(x;\xi)} = \frac{G(x;\theta)}{\bar{G}(x;\theta)} = \frac{G(x;\theta)}{1-G(x;\theta)} = \frac{x}{\theta-x}$, the cumulative distribution function of a new distribution called Weighted Exponential - Uniform $(WE - U)$distribution with three parameters can be obtained as:

$$F_{WE-U}(x;\alpha,\lambda,\theta) = \frac{1}{\alpha}\left[(\alpha+1)\left(1 - e^{-\frac{\lambda x}{\theta-x}}\right) + e^{-\frac{\lambda(\alpha+1)x}{\theta-x}} - 1\right]$$

$$= 1 - \frac{1}{\alpha} e^{-\frac{\lambda x}{\theta-x}} \left(\alpha + 1 - e^{-\frac{\lambda(\alpha+1)x}{\theta-x}}\right) \quad \dots (24)$$

and the corresponding pdf will be:

$$f_{WE-U}(x;\alpha,\lambda,\theta) = \frac{\lambda(\alpha+1)\theta}{\alpha(\theta-x)^2}\ e^{-\frac{\lambda x}{\theta-x}} \left(1 - e^{-\frac{\alpha\lambda x}{\theta-x}}\right);0 < x < \theta < \infty \quad \dots (25)$$

Where $\alpha > 0$ is the shape parameter and $\lambda, \theta > 0$ are the scale parameters and according to equation (21), the pdf can be expressed as:

$$f_{WE-U}(x;\alpha,\lambda,\theta) = \frac{\alpha+1}{\alpha\theta} \sum_{k,j=0}^{\infty} \frac{(-1)^k \lambda^{k+1} \Gamma(k+j+2)}{k!\ j!\ \Gamma(k+2)} \left(\frac{x}{\theta}\right)^{k+j} \left(1 - \sum_{k,j=0}^{\infty} \frac{(-1)^k \alpha^k \lambda^k \Gamma(k+j)}{k!\ j!\ \Gamma(k)} \left(\frac{x}{\theta}\right)^{k+j}\right)$$

the estimates of reliability and hazard functions of $WE - U$ distribution at time (t), respectively, are given by:

$$R_{WE-U}(t;\alpha,\lambda,\theta)=\frac{1}{\alpha}e^{-\frac{\lambda t}{\theta-t}}\left(\alpha+1-e^{-\frac{\lambda(\alpha+1)\,t}{\theta-t}}\right) \qquad \ldots(26)$$

$$h_{WE-U}(t;\alpha,\lambda,\theta)=\frac{\lambda(\alpha+1)\,\theta\,\left(1-e^{-\frac{\alpha\lambda t}{\theta-t}}\right)}{(\theta-t)^2\left(\alpha+1-e^{-\frac{\lambda(\alpha+1)\,t}{\theta-t}}\right)} \qquad \ldots(27)$$

Now, to estimate the three unknown parameters α,λ,θ by maximum likelihood estimation method, consider that $\underline{x}=(x_1,x_2,\ x_3,\ldots,x_n)$ be a random sample of the $WE-U$ distribution defined by (25). The likelihood and natural-log likelihood functions for the given sample observations are defined, respectively, by:

$$L(\alpha,\lambda,\theta|\underline{x})=\frac{\lambda^n\,(\alpha+1)^n\,\theta^n}{\alpha^n\prod_{i=1}^{n}(\theta-x_i)^2}\;e^{-\lambda\sum_{i=1}^{n}\frac{x_i}{\theta-x_i}}\prod_{i=1}^{n}\left(1-e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}\right) \qquad \ldots(28)$$

$$\ell_{WE-U}=ln\,L(\alpha,\lambda,\theta|\underline{x})=n\,ln\,\lambda+n\,ln(\alpha+1)+n\,ln\,\theta-n\,ln\,\alpha-2\sum_{i=1}^{n}ln(\theta-x_i)-\lambda\sum_{i=1}^{n}\frac{x_i}{\theta-x_i}+\sum_{i=1}^{n}ln\left(1-e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}\right) \qquad \ldots(29)$$

The maximum likelihood estimates (MLEs) of the above three parameters are the solution of the first partial derivatives of ℓ_{WE-U} equation (29), with respect to that parameters. Since there are no closed forms of the solutions, iterative approximation techniques such as, Newton–Raphson iterative technique can be used to obtain the MLEs. In Newton-Raphson method, the solution of the likelihood equation, at iteration $(h+1)$ is obtained through the following iterative process,

$$\begin{bmatrix}\hat{\alpha}\\ \hat{\lambda}\\ \hat{\theta}\end{bmatrix}^{(h+1)}=\begin{bmatrix}\hat{\alpha}\\ \hat{\lambda}\\ \hat{\theta}\end{bmatrix}^{(h)}-\mathrm{J}_{(h)}^{-1}\begin{bmatrix}\frac{\partial\ell_{WE-U}}{\partial\alpha}\\ \frac{\partial\ell_{WE-U}}{\partial\lambda}\\ \frac{\partial\ell_{WE-U}}{\partial\theta}\end{bmatrix}^{(h)}\quad;\ h=0,1,2,\ldots$$

Where,

$$J_{(h)}=\begin{bmatrix}\frac{\partial^2\ell_{WE-U}}{\partial\alpha^2} & \frac{\partial^2\ell_{WE-U}}{\partial\alpha\,\partial\lambda} & \frac{\partial^2\ell_{WE-U}}{\partial\alpha\,\partial\theta}\\ \frac{\partial^2\ell_{WE-U}}{\partial\lambda\,\partial\alpha} & \frac{\partial^2\ell_{WE-U}}{\partial\lambda^2} & \frac{\partial^2\ell_{WE-U}}{\partial\lambda\,\partial\theta}\\ \frac{\partial^2\ell_{WE-U}}{\partial\theta\,\partial\alpha} & \frac{\partial^2\ell_{WE-U}}{\partial\theta\,\partial\lambda} & \frac{\partial^2\ell_{WE-U}}{\partial\theta^2}\end{bmatrix}^{(h)}$$ is a Jacobean matrix at iteration (h)

$$\frac{\partial\ell_{WE-U}}{\partial\alpha}=\frac{n}{\alpha+1}-\frac{n}{\alpha}+\lambda\sum_{i=1}^{n}\frac{\frac{x_i}{\theta-x_i}\,e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}}{1-e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}}$$

$$\frac{\partial\ell_{WE-U}}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}\frac{x_i}{\theta-x_i}+\alpha\sum_{i=1}^{n}\frac{\frac{x_i}{\theta-x_i}\,e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}}{1-e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}}$$

$$\frac{\partial\ell_{WE-U}}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}\frac{2}{\theta-x_i}+\lambda\sum_{i=1}^{n}\frac{x_i}{(\theta-x_i)^2}-\sum_{i=1}^{n}\frac{\frac{\alpha\lambda x_i}{(\theta-x_i)^2}\,e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}}{1-e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}}$$

$$\frac{\partial^2\ell_{WE-U}}{\partial\alpha^2}=\frac{n}{\alpha^2}-\frac{n}{(\alpha+1)^2}-\lambda^2\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta-x_i}\right)^2e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}}{\left(1-e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}\right)^2}$$

$$\frac{\partial^2\ell_{WE-U}}{\partial\lambda^2}=-\frac{n}{\lambda^2}-\alpha^2\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta-x_i}\right)^2e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}}{\left(1-e^{-\frac{\alpha\lambda x_i}{\theta-x_i}}\right)^2}$$

$$\frac{\partial^2 \ell_{WE-U}}{\partial \theta^2} = -\frac{n}{\theta^2} + \sum_{i=1}^{n} \frac{2}{(\theta - x_i)^2}\left[1 - \frac{\lambda x_i}{\theta - x_i}\right] - \sum_{i=1}^{n} \frac{\frac{\alpha\lambda x_i}{(\theta - x_i)^3} e^{-\frac{\alpha\lambda x_i}{\theta - x_i}}\left[\frac{\alpha\lambda x_i}{\theta - x_i} + 2\left(e^{-\frac{\alpha\lambda x_i}{\theta - x_i}} - 1\right)\right]}{\left(1 - e^{-\frac{\alpha\lambda x_i}{\theta - x_i}}\right)^2}$$

$$\frac{\partial^2 \ell_{WE-U}}{\partial \alpha\, \partial \lambda} = \frac{\partial^2 \ell_{WE-U}}{\partial \lambda\, \partial \alpha} = \sum_{i=1}^{n} \frac{\frac{x_i}{\theta - x_i} e^{-\frac{\alpha\lambda x_i}{\theta - x_i}}\left[1 - e^{-\frac{\alpha\lambda x_i}{\theta - x_i}} - \frac{\alpha\lambda x_i}{\theta - x_i}\right]}{\left(1 - e^{-\frac{\alpha\lambda x_i}{\theta - x_i}}\right)^2}$$

$$\frac{\partial^2 \ell_{WE-U}}{\partial \alpha\, \partial \theta} = \frac{\partial^2 \ell_{WE-U}}{\partial \theta\, \partial \alpha} = \lambda \sum_{i=1}^{n} \frac{\frac{x_i}{(\theta - x_i)^2} e^{-\frac{\alpha\lambda x_i}{\theta - x_i}}\left[\frac{\alpha\lambda x_i}{\theta - x_i} - 1 + e^{-\frac{\alpha\lambda x_i}{\theta - x_i}}\right]}{\left(1 - e^{-\frac{\alpha\lambda x_i}{\theta - x_i}}\right)^2}$$

$$\frac{\partial^2 \ell_{WE-U}}{\partial \lambda\, \partial \theta} = \frac{\partial^2 \ell_{WE-U}}{\partial \theta\, \partial \lambda} = \sum_{i=1}^{n} \frac{x_i}{(\theta - x_i)^2} + \alpha \sum_{i=1}^{n} \frac{\frac{x_i}{(\theta - x_i)^2} e^{-\frac{\alpha\lambda x_i}{\theta - x_i}}\left[\frac{\alpha\lambda x_i}{\theta - x_i} - 1 + e^{-\frac{\alpha\lambda x_i}{\theta - x_i}}\right]}{\left(1 - e^{-\frac{\alpha\lambda x_i}{\theta - x_i}}\right)^2}$$

When the convergence occurs between iteration $(h+1)$ and (h) , i.e. the absolute difference between two successive iterations is less than pre-specified error tolerance, $\varepsilon > 0$, then the current $\hat{\alpha}^{(h+1)}$, $\hat{\lambda}^{(h+1)}$ and $\hat{\theta}^{(h+1)}$ represent the MLEs of α , λ and θ via NR algorithm which we refer to as, $\hat{\alpha}_{ML}$, $\hat{\lambda}_{ML}$ and $\hat{\theta}_{ML}$.

Then, according to an invariant property of the ML estimator, the reliability and hazard functions at mission time (t) of the $WE-U$ distribution can be obtained, respectively, by replacing α , λ and θ in equations (26) and (27) by their ML estimates as:

$$\hat{R}_{WE-U}^{ML}(t;\alpha,\lambda,\theta) = \frac{1}{\hat{\alpha}_{ML}} e^{-\frac{\hat{\lambda}_{ML}\, t}{\hat{\theta}_{ML} - t}}\left(\hat{\alpha}_{ML} + 1 - e^{-\frac{\hat{\lambda}_{ML}(\hat{\alpha}_{ML}+1)t}{\hat{\theta}_{ML} - t}}\right)$$

$$\hat{h}_{WE-U}^{ML}(t;\alpha,\lambda,\theta) = \frac{\hat{\lambda}_{ML}(\hat{\alpha}_{ML}+1)\hat{\theta}_{ML}\left(1 - e^{-\frac{\hat{\alpha}_{ML}\hat{\lambda}_{ML}\, t}{\hat{\theta}_{ML} - t}}\right)}{\left(\hat{\theta}_{ML} - t\right)^2\left(\hat{\alpha}_{ML} + 1 - e^{-\frac{\hat{\lambda}_{ML}(\hat{\alpha}_{ML}+1)t}{\hat{\theta}_{ML} - t}}\right)}$$

Weighted Exponential - Kumaraswamy Distribution

Consider the parent distribution is Kumaraswamy distribution with density and cumulative distribution functions, respectively, given by [15]:

$$g_k(x;a,b) = abx^{a-1}(1-x^a)^{b-1} \quad ;\ 0 < x < 1\ ;\ a,b > 0 \quad \dots (30)$$

$$G_k(x;a,b) = 1-(1-x^a)^b \quad ;\ 0 < x < 1\ ;\ a,b > 0 \quad \dots (31)$$

Now, depending on equation (10), $\frac{G(x;\xi)}{\bar{G}(x;\xi)} = \frac{G(x;a,b)}{\bar{G}(x;a,b)} = \frac{G(x;a,b)}{1-G(x;a,b)} = (1-x^a)^{-b} - 1$, the cumulative distribution function of a new distribution called Weighted Exponential - Kumaraswamy $(WE-K)$distribution with four parameters can be obtained as:

$$F_{WE-k}(x;\alpha,\lambda,a,b) = 1 - \frac{1}{\alpha} e^{-\lambda\left[(1-x^a)^{-b}-1\right]}\left(\alpha + 1 - e^{-\lambda(\alpha+1)\left[(1-x^a)^{-b}-1\right]}\right) \quad \dots (32)$$

and the corresponding pdf will be:

$$f_{WE-K}(x;\alpha,\lambda,a,b) = \frac{\lambda(\alpha+1)a\, bx^{a-1}}{\alpha(1-x^a)^{b+1}}\ e^{-\lambda\left[(1-x^a)^{-b}-1\right]}\left(1 - e^{-\lambda\alpha\left[(1-x^a)^{-b}-1\right]}\right) \quad \dots (33)$$

where $0 < x < 1$, $\lambda > 0$ is the scale parameter and $\alpha, a, b > 0$ are the shape parameters and according to equation (21), the pdf can be expressed as:

$$f_{WE-K}(x;\alpha,\lambda,a,b) = \frac{\alpha+1}{\alpha} abx^{a-1}(1-x^a)^{b-1} \sum_{k,j=0}^{\infty} \frac{(-1)^k \lambda^{k+1}\,\Gamma(k+j+2)}{k!\ j!\ \Gamma(k+2)} \left(1-(1-x^a)^b\right)^{k+j} \left(1 - \sum_{k,j=0}^{\infty} \frac{(-1)^k \alpha^k \lambda^k\,\Gamma(k+j)}{k!\ j!\ \Gamma(k)} \left(1-(1-x^a)^b\right)^{k+j}\right)$$

The reliability and hazard functions of $WE-K$ distribution, respectively, are given by:

$$R_{WE-K}(t;\alpha,\lambda,a,b) = \frac{1}{\alpha} e^{-\lambda[(1-t^a)^{-b}-1]} \left(\alpha + 1 - e^{-\lambda(\alpha+1)[(1-t^a)^{-b}-1]}\right) \quad \dots (34)$$

$$h_{WE-K}(t;\alpha,\lambda,a,b) = \frac{\lambda(\alpha+1)a\,bt^{a-1}\left(1 - e^{-\lambda\alpha[(1-t^a)^{-b}-1]}\right)}{(1-t^a)^{b+1}\left(\alpha+1-e^{-\lambda(\alpha+1)[(1-t^a)^{-b}-1]}\right)} \quad \dots (35)$$

Now, the MLEs of α, λ, a, b are the solutions of the first partial derivatives of the natural-log likelihood function ℓ_{WE-K} with respect to those parameters where the likelihood and natural-log likelihood functions for equation (33) are defined, respectively, by:

$$L(\alpha,\lambda,a,b|\underline{x}) = \frac{\lambda^n(\alpha+1)^n a^n b^n \prod_{i=1}^{n} x_i^{a-1}}{\alpha^n \prod_{i=1}^{n}(1-x_i^{a})^{b+1}} e^{-\lambda\sum_{i=1}^{n}[(1-x_i^{a})^{-b}-1]} \prod_{i=1}^{n}\left(1 - e^{-\alpha\lambda[(1-x_i^{a})^{-b}-1]}\right) \quad \dots (36)$$

$$\ell_{WE-K} = ln\,L(\alpha,\lambda,a,b|\underline{x}) = n\,ln\,\lambda + n\,ln(\alpha+1) + n\,ln\,a + n\,ln\,b + (a-1)\sum_{i=1}^{n} ln\,x_i - n\,ln\,\alpha - (b+1)\sum_{i=1}^{n} ln(1-x_i^{a}) - \lambda\sum_{i=1}^{n}[(1-x_i^{a})^{-b}-1] + \sum_{i=1}^{n} ln\left(1-e^{-\alpha\lambda[(1-x_i^{a})^{-b}-1]}\right) \quad \dots (37)$$

Now, since there are no closed forms of the solutions, Newton–Raphson iterative technique, can be used to obtain the MLEs as,

$$\begin{bmatrix}\hat{\alpha}\\ \hat{\lambda}\\ \hat{a}\\ \hat{b}\end{bmatrix}^{(h+1)} = \begin{bmatrix}\hat{\alpha}\\ \hat{\lambda}\\ \hat{a}\\ \hat{b}\end{bmatrix}^{(h)} - J_{(h)}^{-1}\begin{bmatrix}\frac{\partial \ell_{WE-K}}{\partial \alpha}\\ \frac{\partial \ell_{WE-K}}{\partial \lambda}\\ \frac{\partial \ell_{WE-K}}{\partial a}\\ \frac{\partial \ell_{WE-K}}{\partial b}\end{bmatrix}^{(h)} \quad ;\ h = 0,1,2,\dots$$

Where,

$$J_{(h)} = \begin{bmatrix}
\frac{\partial^2 \ell_{WE-K}}{\partial \alpha^2} & \frac{\partial^2 \ell_{WE-K}}{\partial \alpha\,\partial \lambda} & \frac{\partial^2 \ell_{WE-K}}{\partial \alpha\,\partial a} & \frac{\partial^2 \ell_{WE-K}}{\partial \alpha\,\partial b}\\
\frac{\partial^2 \ell_{WE-K}}{\partial \lambda\,\partial \alpha} & \frac{\partial^2 \ell_{WE-K}}{\partial \lambda^2} & \frac{\partial^2 \ell_{WE-K}}{\partial \lambda\,\partial a} & \frac{\partial^2 \ell_{WE-K}}{\partial \lambda\,\partial b}\\
\frac{\partial^2 \ell_{WE-K}}{\partial a\,\partial \alpha} & \frac{\partial^2 \ell_{WE-K}}{\partial a\,\partial \lambda} & \frac{\partial^2 \ell_{WE-K}}{\partial a^2} & \frac{\partial^2 \ell_{WE-K}}{\partial a\,\partial b}\\
\frac{\partial^2 \ell_{WE-K}}{\partial b\,\partial \alpha} & \frac{\partial^2 \ell_{WE-K}}{\partial b\,\partial \lambda} & \frac{\partial^2 \ell_{WE-K}}{\partial b\,\partial a} & \frac{\partial^2 \ell_{WE-K}}{\partial b^2}
\end{bmatrix}^{(h)}$$

$$\frac{\partial \ell_{WE-K}}{\partial \alpha} = \frac{n}{\alpha+1} - \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{\lambda[(1-x_i^a)^{-b}-1]\,e^{-\alpha\lambda[(1-x_i^a)^{-b}-1]}}{1-e^{-\alpha\lambda[(1-x_i^a)^{-b}-1]}}$$

$$\frac{\partial \ell_{WE-K}}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}[(1-x_i^a)^{-b}-1] + \sum_{i=1}^{n} \frac{\alpha\,[(1-x_i^a)^{-b}-1]\,e^{-\alpha\lambda[(1-x_i^a)^{-b}-1]}}{1-e^{-\alpha\lambda[(1-x_i^a)^{-b}-1]}}$$

$$\frac{\partial \ell_{WE-K}}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \ln x_i + (b-1)\sum_{i=1}^{n} \frac{x_i^a \ \ln x_i}{1-x_i^a} - \sum_{i=1}^{n} \lambda b(1-x_i^a)^{-(b+1)}\, x_i^a \ \ln x_i + \sum_{i=1}^{n} \frac{\alpha\lambda b(1-x_i^a)^{-(b+1)}\, x_i^a \ \ln x_i\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}$$

$$\frac{\partial \ell_{WE-K}}{\partial b} = \frac{n}{b} - \sum_{i=1}^{n} \ln(1-x_i^a) + \sum_{i=1}^{n} \lambda b(1-x_i^a)^{-b} \ln(1-x_i^a) - \sum_{i=1}^{n} \frac{\alpha\lambda(1-x_i^a)^{-b} \ln(1-x_i^a)\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}$$

$$\frac{\partial^2 \ell_{WE-K}}{\partial \alpha^2} = \frac{n}{\alpha^2} - \frac{n}{(\alpha+1)^2} - \sum_{i=1}^{n} \frac{\left(\lambda((1-x_i^a)^{-b}-1)\right)^2 e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} \left[\frac{e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} + 1\right]$$

$$\frac{\partial^2 \ell_{WE-K}}{\partial \lambda^2} = -\frac{n}{\lambda^2} - \sum_{i=1}^{n} \frac{\alpha^2[(1-x_i^a)^{-b}-1]^2\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} \left[1 + \frac{e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}\right]$$

$$\begin{aligned}\frac{\partial^2 \ell_{WE-K}}{\partial a^2} = {} & (b-1)\sum_{i=1}^{n} \frac{x_i^a\, (\ln x_i)^2}{1-x_i^a}\left[1+\frac{x_i^a}{1-x_i^a}\right] - \frac{n}{a^2} - \sum_{i=1}^{n} \lambda b(1-x_i^a)^{-(b+1)}\, x_i^a\, (\ln x_i)^2 \left[\frac{b+1}{1-x_i^a}+1\right] \\ & - \sum_{i=1}^{n} \frac{\alpha^2\lambda^2 b^2(1-x_i^a)^{-2(b+1)}\, (x_i^a \ln x_i)^2\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} \left[1 + \frac{e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}\right] \\ & + \sum_{i=1}^{n} \frac{\alpha\lambda(1-x_i^a)^{-(b+1)}\, x_i^a\, (\ln x_i)^2\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} \left[\frac{b+1}{1-x_i^a}+1\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ell_{WE-K}}{\partial b^2} = {} & -\frac{n}{b^2} - \sum_{i=1}^{n} \lambda(1-x_i^a)^{-b}\, [\ln(1-x_i^a)]^2 \\ & + \sum_{i=1}^{n} \frac{\alpha\lambda(1-x_i^a)^{-b}\ [\ln(1-x_i^a)]^2\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} \left[1 - \alpha\lambda(1-x_i^a)^{-b}\left(1 + \frac{e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}\right)\right]\end{aligned}$$

$$\frac{\partial^2 \ell_{WE-K}}{\partial \alpha \partial \lambda} = \frac{\partial^2 \ell_{WE-K}}{\partial \lambda \partial \alpha} = \sum_{i=1}^{n} \frac{[(1-x_i^a)^{-b}-1]\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} \left[1 - \alpha\lambda[(1-x_i^a)^{-b}-1]\left(1 + \frac{e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}\right)\right]$$

$$\frac{\partial^2 \ell_{WE-K}}{\partial \alpha \partial a} = \frac{\partial^2 \ell_{WE-K}}{\partial a \partial \alpha} = \sum_{i=1}^{n} \frac{\lambda b x_i^a \ln x_i\, (1-x_i^a)^{-(b+1)}\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} \left[1 - \alpha\lambda[(1-x_i^a)^{-b}-1]\left(\frac{e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} + 1\right)\right]$$

$$\frac{\partial^2 \ell_{WE-K}}{\partial \alpha \partial b} = \frac{\partial^2 \ell_{WE-K}}{\partial b \partial \alpha} = \sum_{i=1}^{n} \frac{\lambda(1-x_i^a)^{-b} \ln(1-x_i^a)^{-b}\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} \left[\alpha\lambda((1-x_i^a)^{-b}-1)\left(\frac{e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\ e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} + 1\right) - 1\right]$$

$$\frac{\partial^2 \ell_{WE-K}}{\partial\lambda\partial a} = \frac{\partial^2 \ell_{WE-K}}{\partial a\partial\lambda} = \sum_{i=1}^{n} \frac{\alpha b(1-x_i^a)^{-(b+1)}\, x_i^a \, ln\, x_i \, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} \left[1 - \alpha\lambda((1-x_i^a)^{-b}-1)\left(1+\frac{e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}\right)\right] - \sum_{i=1}^{n} \frac{b\, x_i^a \, ln\, x_i}{(1-x_i^a)^{b+1}}$$

$$\frac{\partial^2 \ell_{WE-K}}{\partial\lambda\partial b} = \frac{\partial^2 \ell_{WE-K}}{\partial b\partial\lambda} = \sum_{i=1}^{n} \frac{ln(1-x_i^a)}{(1-x_i^a)^b} + \sum\nolimits_{i=1}^{n} \frac{\alpha(1-x_i^a)^{-b} ln(1-x_i^a)\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} \left[\alpha\lambda((1-x_i^a)^{-b}-1)\left(1+\frac{e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}\right) - 1\right]$$

$$\frac{\partial^2 \ell_{WE-K}}{\partial b\partial a} = \frac{\partial^2 \ell_{WE-K}}{\partial a\partial b} = \sum_{i=1}^{n} \frac{x_i^a \, ln\, x_i}{1-x_i^a} + \sum_{i=1}^{n} \frac{\lambda x_i^a ln\, x_i[b\, ln(1-x_i^a)-1]}{(1-x_i^a)^{b+1}} + \sum_{i=1}^{n} \frac{\alpha^2\lambda^2 b\,(1-x_i^a)^{-2(b+1)}\, x_i^a \, ln\, x_i \, ln(1-x_i^a)\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}} \left[1+\frac{e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}\right] + \sum_{i=1}^{n} \frac{\alpha\lambda(1-x_i^a)^{-(b+1)} x_i^a \, ln\, x_i \, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}{1-\, e^{-\alpha\lambda\left[(1-x_i^a)^{-b}-1\right]}}[1-b ln(1-x_i^a)]$$

When the convergence occurs between iteration $(h+1)$ and (h), i.e. the absolute difference between two successive iterations is less than pre-specified error tolerance, $\varepsilon > 0$, then the current $\hat{\alpha}^{(h+1)}$, $\hat{\lambda}^{(h+1)}$, $\hat{a}^{(h+1)}$ and $\hat{b}^{(h+1)}$ represent the MLEs of α, λ, a and b via NR algorithm which we refer to as, $\hat{\alpha}_{ML}$, $\hat{\lambda}_{ML}$, $\hat{a}_{ML}$ and $\hat{b}_{ML}$.

Then, according to an invariant property of the ML estimator, the estimates of reliability and hazard functions at mission time (t) can be obtained, respectively, by replacing α, λ, a and b in equations (34) and (35) by their ML estimates.

Concluding Remarks

Based on the idea of Bourguignon *et al*., a new family to weighted exponential distribution named weighted exponential $-G$ family of distributions has been presented. We concluded that the probability density function of weighted exponential $-G$ family can be expressed as an infinite linear combination of G distribution. Further, two examples of weighted exponential $-G$ family of distributions "weighted exponential - uniform distribution and weighted exponential - Kumaraswamy distribution" are discussed along with the maximum likelihood estimation of their parameters.

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