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Ritz Method for the Analysis of Statically Indeterminate Euler – Bernoulli Beam

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Original Research Article

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Abstract: The Ritz method was used in this paper for the flexural analysis of a statically indeterminate Euler – Bernoulli beam with a prismatic cross section. The beam considered was a propped cantilever of length, l, fixed at $x = 0$, and simply supported at $x = l$; and carrying a linearly distributed transverse load on the longitudinal axis. Two cases of coordinate (basis) functions were studied. In the first case, the basis functions were constructed to satisfy the deflection boundary conditions, but not the force boundary conditions. In the second case, the basis functions were constructed to satisfy all the boundary conditions. It was found that the stiffness equations formed with the basis functions that satisfied all the boundary conditions gave the exact solutions for deflection, bending moments and shear force distributions along the beam's longitudinal axis. The effectiveness of the Ritz method for solving statically indeterminate Euler Bernoulli beam flexure problems was thus highlighted.

Keywords: Ritz method, Euler – Bernoulli beam, stiffness equations, statically indeterminate beam, deflection, bending moment distribution, shears force distribution.

INTRODUCTION

Beams are structural members that carry loads applied transversely to the longitudinal axis by the development of bending moments. They are used in buildings, bridges, machines, aerospace structures and naval structures [1, 2].

Several theories are used to describe the behaviour of beams. Some of these are: Euler – Bernoulli beam theory [3], Mindlin beam theory, Timoshenko beam theory [4] and Vlasov beam theory. Euler – Bernoulli beam theory has been found to be suitable for slender beams where the flexural behaviour governs the structural behaviour [5,6]. For deep and moderately thick beams, shear deformation plays a significant role in the structural behaviour, and the Timoshenko beam theory is used [4].

The main defect of the Euler – Bernoulli beam theory is that it is inaccurate for deep (thick) beams. Thick beams are ones for which the thickness is not negligible compared to the length (longitudinal dimension). A more accurate beam theory or a complete three dimensional (3D) solid mechanics approach is used to analyse thick beams. Timoshenko beam theory can be considered an extension of the Euler – Bernoulli beam theory in that it considers the effect of shear deformation, which is disregarded in the Euler Bernoulli beam theory.

The fundamental assumptions of the Timoshenko beam theory are:

- The longitudinal axis of the unloaded beam is straight.
- Loads are applied transversely to the longitudinal axis.
- The deformations and strains are small.
- Hooke's law of linear elasticity can be used for the stress – strain relations.
- Plane cross – sections which are initially normal to the longitudinal axis will remain plane after bending deformation.

This paper adopts the Euler – Bernoulli beam theory. Several methods are used in the analysis of the Euler Bernoulli beam to determine the internal bending moments and shear force distribution.

For statically indeterminate beams, the methods found in the literature include:

- Force methods
- Flexibility methods

- Energy methods [7, 8]

Fundamental assumptions of the Euler – Bernoulli beam theory

The assumptions of the Euler – Bernoulli beam theory are as follows: [9]

- The beam is prismatic and has a straight centroidal axis, which is defined as the x – axis.
- The beam's cross-section has an axis of symmetry.
- All transverse loadings act in the plane of symmetry.
- Plane sections perpendicular to the centroidal axis remain plane after bending deformation.
- The beam material is elastic, isotropic and homogeneous.
- Transverse deflections are small.

METHODOLOGY

The statically indeterminate Euler – Bernoulli beam problem shown in Figure 1, was considered.

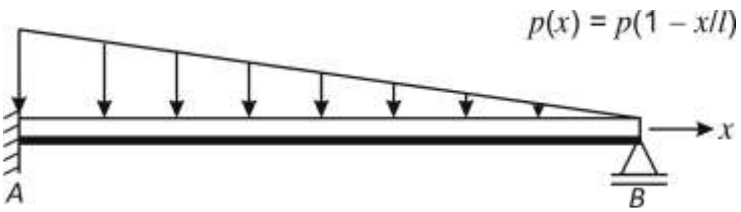

Fig-1: A statically indeterminate Euler – Bernoulli beam under linear load distribution

The origin of coordinates is chosen at the fixed support, A. The boundary conditions are:

$w(0) = 0$ (1); $w'(0) = 0$ (2); $w(l) = 0$ (3); $M(l) = 0$ (4); $w''(l) = 0$ (5)

The deflection shape functions are constructed from third degree and fourth degree polynomials as follows:

$$w(x) = c_1\left(a_0 + a_1\frac{x}{l} + a_2\frac{x^2}{l^2} + a_3\frac{x^3}{l^3}\right) + c_2\left(b_0 + b_1\frac{x}{l} + b_2\frac{x^2}{l^2} + b_3\frac{x^3}{l^3} + b_4\frac{x^4}{l^4}\right)$$

$$= c_1N_1(x) + c_2N_2(x) \tag{7}$$

where c_1 and c_2 are the unknown generalized deflections, N_1 and N_2 are the shape functions.

$$w'(x) = c_1N_1' + c_2N_2' \tag{8}$$

$$w'(x) = c_1\left(a_1\frac{1}{l} + 2a_2\frac{x}{l^2} + 3a_3\frac{x^2}{l^3}\right) + c_2\left(b_1\frac{1}{l} + 2b_2\frac{x}{l^2} + 3b_3\frac{x^2}{l^3} + 4b_4\frac{x^3}{l^4}\right) \tag{9}$$

$$w''(x) = c_1\left(2a_2\frac{1}{l^2} + 6a_3\frac{x}{l^3}\right) + c_2\left(2b_2\frac{1}{l^2} + 6b_3\frac{x}{l^3} + 12b_4\frac{x^2}{l^4}\right) \tag{10}$$

$$w(0) = 0 \Rightarrow a_0 = b_0 = 0 \tag{11}$$

$$w'(0) = 0 \Rightarrow a_1 = b_1 = 0 \tag{12}$$

$$w(l) = 0 \Rightarrow a_2 + a_3 = 0 \tag{13}$$

$$b_2 + b_3 + b_4 = 0 \tag{14}$$

$$w''(l) = 0$$

$$\frac{2a_2}{l^2} + \frac{6a_3}{l^2} = 0 \tag{15}$$

$$\frac{2b_2}{l^2} + \frac{6b_3}{l^2} + \frac{12b_4}{l^2} = 0 \tag{16}$$

$$2a_2 + 6a_3 = 0 \tag{17}$$

$$2b_2 + 6b_3 + 12b_4 = 0 \tag{18}$$

Solving, using $w(0) = 0$, $w'(0) = 0$, $w(l) = 0$

$$a_2 = -a_3 = 1 \quad (19)$$

$$b_2 = b_4 = 1 \quad b_3 = -2 \quad (20)$$

$$w(x) = c_1\left(\frac{x^2}{l^2} - \frac{x^3}{l^3}\right) + c_2\left(\frac{x^2}{l^2} - 2\frac{x^3}{l^3} + \frac{x^4}{l^4}\right) = c_1N_1 + c_2N_2 \quad (21)$$

$$N = \left(\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)^3 ; \left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4\right) \quad (22)$$

$$N'' = \frac{1}{l^2}\left(\left(2 - 6\frac{x}{l}\right)\left(2 - 12\frac{x}{l} + 12\frac{x^2}{l^2}\right)\right) \quad (23)$$

The stiffness matrix is

$$K = \int_0^l EI\, N''^T\, N''dx \quad (24)$$

Where T denotes the transpose matrix operation, E is the Young's modulus, and I is the moment of inertia.

$$K = \int_0^l EI\begin{pmatrix} N_1'' \\ N_2'' \end{pmatrix}(N_1''\, N_2'')\, dx \quad (25)$$

$$K = EI\int_0^l \begin{pmatrix} (N_1'')^2 & N_1''N_2'' \\ N_2''N_1'' & (N_2'')^2 \end{pmatrix} dx \quad (26)$$

$$K = \frac{EI}{l^4}\int_0^l \left(\begin{pmatrix} \left(2 - \frac{6x}{l}\right) \\ \left(2 - \frac{12x}{l} + \frac{12x^2}{l^2}\right) \end{pmatrix}\left(\left(2 - \frac{6x}{l}\right)\left(2 - \frac{12x}{l} + \frac{12x^2}{l^2}\right)\right)\right) dx \quad (27)$$

$$K = \frac{EI}{l^4}\int_0^l \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} dx \quad (28)$$

$$a_{11} = \left(2 - \frac{6x}{l}\right)^2 = 4 - \frac{24x}{l} + \frac{36x^2}{l^2} \quad (29)$$

$$a_{12} = \left(2 - \frac{6x}{l}\right)\left(2 - \frac{12x}{l} + \frac{12x^2}{l^2}\right) \quad (30)$$

$$a_{12} = 4 - \frac{36x}{l} + \frac{96x^2}{l^2} - \frac{72x^3}{l^3} \quad (31)$$

$$a_{21} = \left(2 - \frac{12x}{l} + \frac{12x^2}{l^2}\right)\left(2 - \frac{6x}{l}\right) \quad (32)$$

$$a_{21} = 4 - \frac{36x}{l} + \frac{96x^2}{l^2} - \frac{72x^3}{l^3} \quad (33)$$

$$a_{22} = \left(2 - \frac{12x}{l} + \frac{12x^2}{l^2}\right)^2 \quad (34)$$

$$a_{22} = 4 - \frac{48x}{l} + \frac{48x^2}{l^2} - \frac{288x^3}{l^3} + \frac{144x^4}{l^4} + \frac{144x^2}{l^2} \quad (35)$$

Evaluating the integrals, we obtain:

$$K = \frac{EIl}{l^4}\begin{pmatrix} 4 & 0 \\ 0 & 0.8 \end{pmatrix} = \frac{EI}{l^3}\begin{pmatrix} 4 & 0 \\ 0 & 0.8 \end{pmatrix} \quad (36)$$

Force (load) matrix

The force matrix is

$$F = \int_0^l p(x)\begin{pmatrix} N_1(x) \\ N_2(x) \end{pmatrix} dx \quad (37)$$

$$F = \int_0^l p_0\left(1 - \frac{x}{l}\right)\begin{pmatrix} \left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)^3 \\ \left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4 \end{pmatrix} dx \quad (38)$$

$$F = p_0\int_0^l \begin{Bmatrix} \left(1 - \frac{x}{l}\right)\left(\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)^3\right) \\ \left(1 - \frac{x}{l}\right)\left(\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4\right) \end{Bmatrix} dx \quad (39)$$

$$F = p_0\int_0^l \begin{pmatrix} \left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4 \\ \left(\frac{x}{l}\right)^2 - 3\left(\frac{x}{l}\right)^3 + 3\left(\frac{x}{l}\right)^4 - \left(\frac{x}{l}\right)^5 \end{pmatrix} dx \quad (40)$$

$$F = p_0 l\begin{pmatrix} 1/30 \\ 1/60 \end{pmatrix} \quad (41)$$

The structure stiffness matrix is

$$\frac{EI}{l^3}\begin{pmatrix} 4 & 0 \\ 0 & 0.8 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = p_0 l\begin{pmatrix} 1/30 \\ 1/60 \end{pmatrix} \quad (42)$$

Solving,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/120 \\ 1/48 \end{pmatrix}\frac{p_0 l^4}{EI} \quad (43)$$

The deflection function becomes:

$$w(x) = \frac{p_0 l^4}{EI}\left(\frac{1}{120}\left(\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)^3\right)\right) + \frac{1}{48}\left(\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4\right) \quad (44)$$

Simplifying, the deflection function is:

$$w(x) = \frac{p_0 l^4}{240EI}\left(7\left(\frac{x}{l}\right)^2 - 12\left(\frac{x}{l}\right)^3 + 5\left(\frac{x}{l}\right)^4\right) \quad (45)$$

The bending moments $M(x)$ and shear force $Q(x)$ distributions are found from the bending moment deflection and the shear force – deflection relations as:

$$M(x) = -EIw'' = \frac{-p_0 l^2}{120}\left(7 - \frac{36x}{l} + \frac{30x^2}{l^2}\right) \quad (46)$$

$$Q(x) = -EIw'''(x) = \frac{p_0 l}{30}\left(8 - \frac{15x}{l}\right) \quad (47)$$

We observe that at $x = l$,

$$M(l) = \frac{-p_0 l^2}{120}(7 - 36 + 30) = \frac{-p_0 l^2}{120} \neq 0 \qquad (48)$$

The boundary condition at $x = l$, i.e. $M(l) = 0$ is violated since the shape functions were not constructed to comply with this boundary condition.

We consider another solution of the problem where the shape functions satisfy all the boundary conditions. Here,

$$w(x) = c_1\left(\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)^3\right) + c_2\left(\left(\frac{x}{l}\right)^3 - \left(\frac{x}{l}\right)^4\right) + c_3\left(\left(\frac{x}{l}\right)^4 - \left(\frac{x}{l}\right)^5\right) = \sum_{i=1}^{3} c_i N_i \qquad (49)$$

where c_1, c_2 and c_3 are the three unknown generalized deflection parameters.
The stiffness matrix is

$$K = EI\int_0^l \begin{pmatrix} \left(\frac{2}{l^2}\right) - \left(\frac{6x}{l^3}\right) \\ \left(\frac{6x}{l^2}\right) - \left(\frac{12x^2}{l^4}\right) \\ \left(\frac{12x^2}{l^4}\right) - \left(\frac{20x^3}{l^5}\right) \end{pmatrix} \left(\left(\left(\frac{2}{l^2}\right) - \left(\frac{6x}{l^3}\right)\right), \left(\left(\frac{6x}{l^2}\right) - \left(\frac{12x^2}{l^4}\right)\right)\left(\left(\frac{12x^2}{l^4}\right) - \left(\frac{20x^3}{l^5}\right)\right)\right) dx \qquad (50)$$

$$K = EI\int_0^l \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} dx \qquad (51)$$

$$n_1 = \frac{2}{l^2} - \frac{6x}{l^3} = \frac{1}{l^2}\left(2 - \frac{6x}{l}\right) \qquad (52)$$

$$n_2 = \frac{6x}{l^2} - \frac{12x^2}{l^4} = \frac{1}{l^2}\left(6x - \frac{12x^2}{l^2}\right) \qquad (53)$$

$$n_3 = \frac{12x^2}{l^4} - \frac{20x^3}{l^5} = \frac{1}{l^2}\left(\frac{12x^2}{l^2} - \frac{20x^3}{l^3}\right) \qquad (54)$$

$$a_{11} = \frac{1}{l^4}\left(2 - \frac{6x}{l}\right)^2 \qquad (55)$$

$$a_{12} = \frac{1}{l^4}\left(2 - \frac{6x}{l}\right)\left(6x - \frac{12x^2}{l^2}\right) = a_{21} \qquad (56)$$

$$a_{13} = \frac{1}{l^4}\left(2 - \frac{6x}{l}\right)\left(\frac{12x^2}{l^2} - \frac{20x^3}{l^3}\right) = a_{31} \qquad (57)$$

$$a_{22} = \frac{1}{l^4}\left(6x - \frac{12x^2}{l^2}\right)^2 \qquad (58)$$

$$a_{33} = \frac{1}{l^4}\left(\frac{12x^2}{l^2} - \frac{20x^3}{l^3}\right)^2 \qquad (59)$$

$$a_{23} = \frac{1}{l^4}\left(6x - \frac{12x^2}{l^2}\right)\left(\frac{12x^2}{l^2} - \frac{20x^3}{l^3}\right) \qquad (60)$$

Evaluation of the integrals in the stiffness matrix yields the stiffness matrix as Equation (61):

$$K = \frac{EI}{l^3}\begin{pmatrix} 4 & 4 & 4 \\ 4 & 24/5 & 26/5 \\ 4 & 26/5 & 208/35 \end{pmatrix} \quad (61)$$

Force (load) vector
The load vector, F is

$$F = \int_0^l p(x)\, N \, dx = p_0 \int_0^l \left(1 - \frac{x}{l}\right) \begin{pmatrix} \left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)^3 \\ \left(\frac{x}{l}\right)^3 - \left(\frac{x}{l}\right)^4 \\ \left(\frac{x}{l}\right)^4 - \left(\frac{x}{l}\right)^5 \end{pmatrix} dx \quad (62)$$

$$F = p_0 \int_0^l \begin{pmatrix} \left(1 - \frac{x}{l}\right)\left(\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)^3\right) \\ \left(1 - \frac{x}{l}\right)\left(\left(\frac{x}{l}\right)^3 - \left(\frac{x}{l}\right)^4\right) \\ \left(1 - \frac{x}{l}\right)\left(\left(\frac{x}{l}\right)^4 - \left(\frac{x}{l}\right)^5\right) \end{pmatrix} dx \quad (63)$$

$$F = p_0 \int_0^l \begin{Bmatrix} \left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4 \\ \left(\frac{x}{l}\right)^3 - 2\left(\frac{x}{l}\right)^4 + \left(\frac{x}{l}\right)^5 \\ \left(\frac{x}{l}\right)^4 - 2\left(\frac{x}{l}\right)^5 + \left(\frac{x}{l}\right)^6 \end{Bmatrix} dx \quad (64)$$

Evaluation of the integrals give:

$$F = p_0 l \begin{pmatrix} 1/30 \\ 1/60 \\ 1/105 \end{pmatrix} \quad (65)$$

RESULTS
The structure stiffness matrix equation is:

$$\frac{EI}{l^3}\begin{pmatrix} 4 & 4 & 4 \\ 4 & 24/5 & 26/5 \\ 4 & 26/5 & 208/35 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = p_0 l \begin{pmatrix} 1/30 \\ 1/60 \\ 1/105 \end{pmatrix} \quad (66)$$

Solving,

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix} \frac{p_0 l^4}{120 EI} \quad (67)$$

Deflection

The solution for deflection is:

$$w(x) = \frac{4p_0 l^4}{120EI}\left(\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)^3\right) - \frac{4p_0 l^4}{120EI}\left(\left(\frac{x}{l}\right)^3 - \left(\frac{x}{l}\right)^4\right) + \frac{p_0 l^4}{120EI}\left(\left(\frac{x}{l}\right)^4 - \left(\frac{x}{l}\right)^5\right) \quad (68)$$

Simplification yields:

$$w(x) = \frac{p_0 l^4}{120EI}\left(4\left(\frac{x}{l}\right)^2 - 8\left(\frac{x}{l}\right)^3 + 5\left(\frac{x}{l}\right)^4 - \left(\frac{x}{l}\right)^5\right) \quad (69)$$

This gives the exact solution for $w(x)$.

Bending moment distribution

The bending moment distribution, $M(x)$ is obtained from the bending moment – deflection relation:

$$M(x) = -EI\,w''(x) \quad (70)$$

Where the primes mean differentiation with respect to x

$$w'(x) = \frac{p_0 l^4}{120EI}\left(\frac{8x}{l^2} - \frac{24x^2}{l^3} + \frac{20x^3}{l^4} - \frac{5x^4}{l^5}\right) \quad (71)$$

$$w''(x) = \frac{p_0 l^4}{120EI}\left(\frac{8}{l^2} - \frac{48x}{l^3} + \frac{60x^2}{l^4} - \frac{20x^3}{l^5}\right) \quad (72)$$

$$w'''(x) = \frac{p_0 l^4}{120EI}\left(\frac{-48}{l^3} + \frac{120x}{l^3} - \frac{60x^2}{l^5}\right) \quad (73)$$

$$M(x) = \frac{-p_0 l^4}{120}\left(\frac{8}{l^2} - \frac{48x}{l^3} + \frac{60x^2}{l^4} - \frac{20x^3}{l^5}\right) \quad (74)$$

$$M(0) = \frac{-8p_0 l^4}{120 l^2} = -\frac{p_0 l^2}{15} \quad (75)$$

$$M(l) = \frac{-p_0 l^4}{120}\left(\frac{8}{l^2} - \frac{48}{l^2} + \frac{60}{l^2} - \frac{20}{l^2}\right) = 0 \quad (76)$$

Shear force distribution $Q(x)$

The shear force distribution is obtained from the shear force – deflection relation:

$$Q(x) = -EI\,w'''(x) \quad (77)$$

$$Q(x) = \frac{-p_0 l^4}{120}\left(\frac{-48}{l^3} + \frac{120x}{l^4} - \frac{60x^2}{l^5}\right) \quad (78)$$

$$Q(0) = \left(\frac{-p_0 l^4}{120}\right)\left(\frac{-48}{l^3}\right) = \frac{48p_0 l}{120} = \frac{2p_0 l}{5} \quad (79)$$

$$Q(l) = \frac{-p_0 l^4}{120}\left(\frac{-48}{l^3} + \frac{120}{l^3} - \frac{60}{l^3}\right) = \frac{-12p_0 l}{120} \quad (80)$$

$$Q(l) = \frac{-p_0 l}{10} \quad (81)$$

DISCUSSION

The Ritz method has been used in this paper to analyse the flexural behaviour of a propped cantilever Euler – Bernoulli beam subject to a linear distribution of transverse load. Two cases of the deflection basis function were investigated. In the first problem considered, deflection basis functions that satisfied some but not all the boundary conditions were used in a two parameter Ritz formulation of the problem to determine the unknown displacement parameters. The bending moment and shear force distributions obtained showed that the bending moment distribution violated the force boundary condition at the propped end ($x = l$).

The second case considered a three parameter Ritz solution of the same problem of a statically indeterminate clamped simply supported Euler – Bernoulli beam (propped cantilever beam). The three basis functions used satisfied all the natural and essential boundary conditions of the beam at $x = 0$, and $x = l$. The stiffness matrix was computed as Equation (61) while the load (force) vector was obtained as Equation (65). The structure stiffness matrix equation in the three parameter Ritz formulation was thus obtained as Equation (66); a system of linear algebraic equations in the three unknown deflection parameters. Equation (66) was solved using the techniques for linear algebraic equations to obtain Equation (67). The displacement was thus found for a three parameter Ritz formulation constructed using three basis (coordinate) functions that satisfied all the boundary conditions as Equation (69). Moment – displacement relation of the Euler – Bernoulli beam theory was used to obtain the bending moment distribution as Equation (74). The bending moment at the clamped end was obtained as Equation (75) while the bending moment at the propped end was found as Equation (76). The force boundary condition at the propped end was thus satisfied by this choice of coordinate (basis) functions.

The shear force distribution along the longitudinal axis of the beam was obtained using the shear force – deflection relation as Equation (78). The shear force at the clamped end was found as Equation (79) while the shear force at the propped end was obtained as Equation (81). The results agreed exactly with the classical solutions.

CONCLUSIONS

From the study the following conclusions are made:

- Ritz method can be effectively used to solve the flexural problem of statically indeterminate Euler – Bernoulli beams.
- The Ritz method depends on the construction of appropriate basis (coordinate) functions that satisfy all the deformation and loading boundary conditions for its effectiveness and accuracy.
- The Ritz method converts the boundary value problem of solving the ordinary differential equation of flexure of Euler – Bernoulli beams subject to the boundary conditions to a problem of linear algebra where the unknown displacement parameters are the unknowns.
- With the use of appropriate basis (coordinate) functions that satisfy all the deformation and loading boundary conditions, the Ritz formulation yields exact solutions to the problem within the theoretical framework of the Euler – Bernoulli beam theory.
- When basis functions that violate some of the boundary conditions are used in the Ritz formulation, the solutions obtained would be in error.
- The use of numerical integration schemes permits the extension of the Ritz method to non-prismatic Euler – Bernoulli beams.

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